Queen Mary University of London

MTH5123 Differential Equations

Formative Assessment: Week 8

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- This Coursework consists of three parts:
  - **I.** Practice problems
  - **II.** Mock Quiz Week 8
- A selection of solutions to the listed problems will be posted on QMPlus by the end of Week 8 and discussed during the tutorials.
- I encourage all students to learn and check their computational answers using math software such as Mathematica, MATLAB, etc. Using numerical software is a fun practice and will help you to visualise your solutions.

I. Practice Problems

A. Sketch the following parametric curves.

1) $\begin{cases} y_1(t) &= t+1 \\ y_2(t) &= 2t-3 \end{cases}$ for $0 \leq t < \infty$.

2) $\begin{cases} y_1(t) &= t^2-1 \\ y_2(t) &= 4t+1 \end{cases}$ for $0 \leq t < \infty$.

3) $\begin{cases} y_1(t) &= 5\cos t \\ y_2(t) &= 4\sin t \end{cases}$ for $0 \leq t \leq 4\pi$.

Hint: This is a revision of content from Calculus.
$y_1(t)$ Connections with our lecture in week 8:
we can consider the parametric equations above are particular solutions to some ODE systems. Here, y_1 and y_2 are variables depending on the independent variable t. Here, the phase plane is the xy plane, which does not include the dimension for t. The arrows on the sketched trajectory should indicate how $y_1(t)$ and $y_2(t)$ changes when t increases.

B. Compute all equilibria of the following systems of ODEs

1)

$$\dot{y_1} = y_1^2 - 4y_2, \quad \dot{y_2} = (y_1+2)y_2.$$

2)

$$\dot{y_1} = y_2^2 + y_1y_2, \quad \dot{y_2} = y_1^2 - 2y_2 - 5y_1 + 2.$$

C. In this exercise we practice graphing parametric curves in both cartesian and polar coordinates.

1) Graph the parametric curve given by $\begin{cases} y_1(t) &= e^{-t}\sin 3t \\ y_2(t) &= e^{-t}\cos 3t \end{cases}$ for $0 \leq t < \infty$.

2) Graph the curve defined by the function $r = 4\sin\theta$ in the cartesian coordinate. *Hint: Multiply both sides by r and use the transformation $y_1 = r\cos\theta$, $y_2 = r\sin\theta$ to rewrite the equation in Cartesian coordinates (y_1, y_2).*

3) Graph $r = \theta$ in the cartesian coordinate. Write 1-2 sentences comparing the graphs of **1)** and **3)**.

D. In weeks 1-2, we learned the logistic equation, which is first order ODE used in biology to model population growth (see our typed lecture notes and Week 1 if you are not familiar with this example). In this model, the change of population size P over time t is governed by a 1st-order non-linear separable differential equation.

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right).$$

Here, r is the per capita growth rate, M is the maximum population size, and both are positive real constant parameters (not variables).

In the following example we consider a system of 2 first-order ODEs which includes the population of another species that is a predator of the first species. Let us call $P(t)$ the population size of the prey species and $K(t)$ the population size of the predator species, this system can be modelled as

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M} - aPK\right), \quad \frac{dK}{dt} = aPK - dK.$$

Here, a is the predation rate, d is the death rate of the predator, and both are real positive parameters (not variables).

1) Compute all equilibria of the above ODE system. Hint: the equilibria might contain some or all parameters, r, M, a, d, which are all positive real numbers.

2) Based on the results in (1), find out the condition that the parameters need to satisfy for obtain an equilibrium point of this dynamical system describing the coexistence of the two species.

3) If one of the two species go extinct, i.e. $P = 0$ or $K = 0$, write down the ODE for the dynamics of the remaining species and compute the corresponding equilibria.

II. Mock Quiz

Train yourself for Coursework 2 by answering Mock Quiz Week 8.