

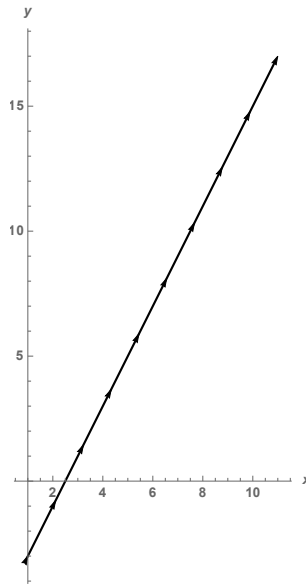
I. Practice Problems

Correction: all coordinates should be y_1, y_2 plane, instead of xy plane in all figures.

A. Sketch the following parametric curves:

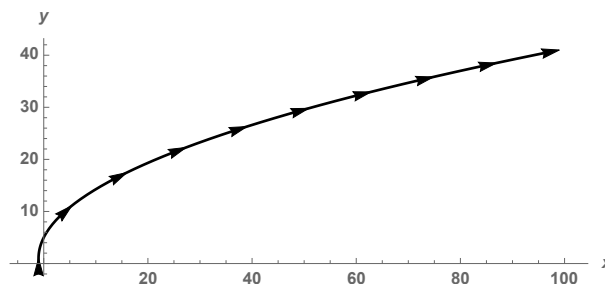
$$1) \begin{cases} y_1(t) = t + 1 \\ y_2(t) = 2t - 3 \end{cases} \text{ for } 0 \leq t < \infty.$$

Solution: Using the first equation to eliminate the parameter, we obtain the equation of the line $y_2 = 2y_1 - 5$. To determine the direction of the parametric curve, by examining various values of t , for example, we see that the arrow on this line points from the third quadrant to the first quadrant.



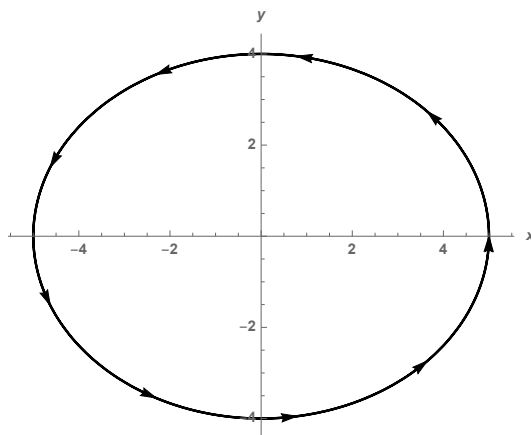
$$2) \begin{cases} y_1(t) = t^2 - 1 \\ y_2(t) = 4t + 1 \end{cases} \text{ for } 0 \leq t < \infty.$$

Solution: Using the *second equation* to eliminate the parameter, we obtain the equation of a parabola $y_1 = \frac{1}{16}(y_2 - 1)^2 - 1$, which opens to the right and has vertex $(-1, 1)$ and crosses the y_2 -axis at ± 5 . To graph the parametric curve for $0 \leq t \leq \infty$, we find $(y_1(0), y_2(0)) = (-1, 1)$ and for $t > 0$, we follow the upper branch of this parabola to infinity.



$$3) \begin{cases} y_1(t) = 5 \cos t \\ y_2(t) = 4 \sin t \end{cases} \text{ for } 0 \leq t \leq 4\pi.$$

Solution: Here, we observe that $y_1/5 = \cos t$ and $y_2/4 = \sin t$, combined with the fact that $\sin^2 t + \cos^2 t = 1$, we have the equation of an ellipse centred at the origin: $\left(\frac{y_1}{5}\right)^2 + \left(\frac{y_2}{4}\right)^2 = 1$, and crossing the y_1 -axis at ± 5 and the y_2 -axis at ± 4 . The direction of the arrow for the parametric curve is anticlockwise (try $t = 0, \pi/2, \pi, \dots$). The parametric curve starts at $(y_1(0), y_2(0)) = (5, 0)$ and goes around the ellipse *twice* before ending at the same point.



B. Compute all equilibria of the following systems of ODEs

1)

$$\dot{y}_1 = y_1^2 - 4y_2, \quad \dot{y}_2 = (y_1 + 2)y_2.$$

Solution. The equilibria can be found when $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$. There are in total 2 equilibria of this ODE system. The right-hand side $\dot{y}_2 = (y_1 + 2)y_2 = 0$ for either $y_1 = -2$ or $y_2 = 0$. For $y_1 = -2$ the right-hand side $\dot{y}_1 = y_1^2 - 4y_2 = 0$ vanishes for $y_2 = 1$, hence we have an equilibrium at the point $(-2, 1)$ in the (x, y) plane. For $y_2 = 0$ the right-hand side $\dot{y}_1 = y_1^2 - 4y_2 = 0$ for $y_1 = y_2 = 0$, which responds to the other equilibrium at the point $(0, 0)$.

2)

$$\dot{y}_1 = y_2^2 + y_1y_2, \quad \dot{y}_2 = y_1^2 - 2y_2 - 5y_1 + 2.$$

Solution. The equilibria can be found when $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$. There are in total 4 equilibria of this ODE system. The right-hand side $\dot{y}_1 = y_2^2 + y_1y_2 = y_2(y_1 + y_2) = 0$ for either $y_2 = -y_1$ or $y_2 = 0$. For $y_1 = y_2$ the right-hand side $\dot{y}_2 = y_1^2 + 2y_1 - 5y_1 + 2 = y_1^2 - 3y_1 + 2 = (y_1 - 1)(y_1 - 2) = 0$ for $y_1 = 1$ or $y_1 = 2$, hence we have two equilibria at the points $(1, -1)$ and $(2, -2)$ in the (y_1, y_2) plane. For $y_2 = 0$ the right-hand side $\dot{y}_1 = y_1^2 - 5y_1 + 2 = 0$ for $y_1 = \frac{5-\sqrt{17}}{2}$ and $y_1 = \frac{5+\sqrt{17}}{2}$, which responds to the other two equilibria at the points, $(\frac{5-\sqrt{17}}{2}, 0)$ and $(\frac{5+\sqrt{17}}{2}, 0)$.

C. In this exercise we practice graphing parametric curves in both cartesian and polar coordinates.

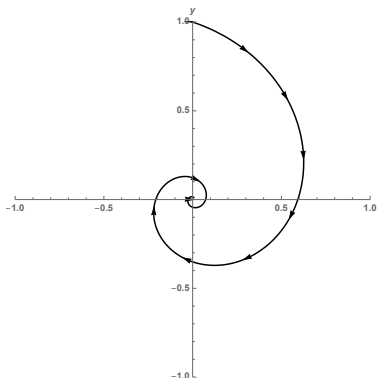
1) Graph the parametric curve given by $\begin{cases} y_1(t) = e^{-t} \sin 3t \\ y_2(t) = e^{-t} \cos 3t \end{cases}$ for $0 \leq t < \infty$.

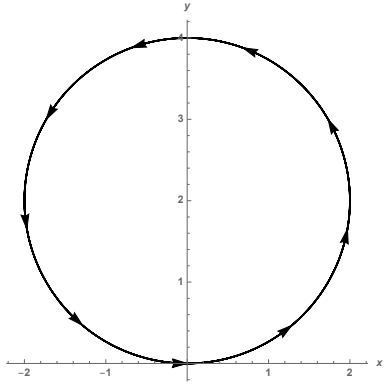
2) Graph the curve defined by the function $r = 4 \sin \theta$ in the cartesian coordinate.
Hint: Multiply both sides by r and use the transformation $y_1 = r \cos \theta$, $y_2 = r \sin \theta$ to rewrite the equation in Cartesian coordinates (y_1, y_2) .

3) Graph $r = \theta$ in the cartesian coordinate. Write 1-2 sentences comparing the graphs of 1) and 3).

Solution:

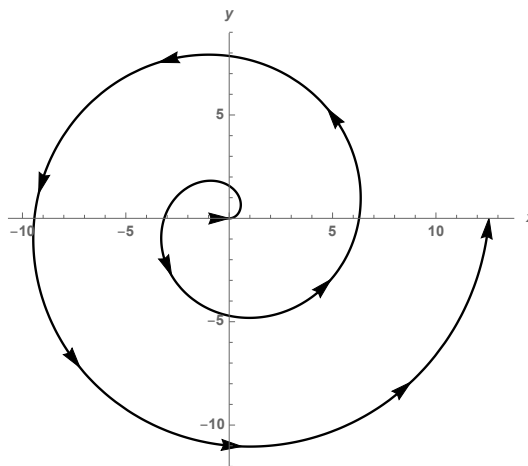
1) $y_1(t) = e^{-t} \sin 3t$, $y_2(t) = e^{-t} \cos 3t$, for $0 \leq t < \infty$ is a clockwise spiral, starting at $(0,1)$, circling toward the origin.





- 2) Using the transformation $y_1 = r \cos \theta$, $y_2 = r \sin \theta$, and multiplying $r = 4 \sin \theta$ by r , gives $r^2 = 4r \sin \theta \Leftrightarrow r^2(\cos^2 \theta + \sin^2 \theta) = 4(r \sin \theta) \Leftrightarrow y_1^2 + y_2^2 = 4y$. Rearranging the terms, we have $y_1^2 + (y_2 - 2)^2 = 4$, which is a circle centred at $(0, 2)$ of radius 2. The parametric curve traverses this curve anticlockwise.

- 3) Using the transformation $y_1 = r \cos \theta$, $y_2 = r \sin \theta$ and $r = \theta$, thus $y_1 = \theta \cos \theta$, $y_2 = \theta \sin \theta$ and $y_1^2 + y_2^2 = \theta^2$. When $\theta = 0$, the parametric curve is at the origin of the Cartesian coordinates (y_1, y_2) . When $\theta = 0, \pi, 2\pi, \dots$, it hits the x coordinate. When $\theta = \frac{1}{2}\pi, \frac{3}{2}\pi, \dots$, it hits the y coordinate. The value of $\sqrt{y_1^2 + y_2^2}$, which is also the distance between the point where the parametric curve hits the x or y coordinate and the origin, linearly increases with θ because $y_1^2 + y_2^2 = \theta^2$. Thus, we can plot this parametric curve as an outward anticlockwise spiral starting at the origin.



Comparing the graphs in (1) and (3), we can see that graph 1) is spiral into the origin, and graph 3) is spiral away from the origin. (Why? This is because the coefficient function before sin and cos are different in these two questions. In (1), it is e^{-t} which decreases with t , and in (3) is θ which linearly increases with θ . This is relevant to our week 9-10 lectures.)

D. In weeks 1-2, we learned the logistic equation, which is first order ODE used in biology to model population growth (see our typed lecture notes and Week1 if you are not familiar with this example). In this model, the change of population size P over time t is governed by a 1st-order non-linear separable differential equation.

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{M} \right).$$

Here, r is the per capita growth rate, M is the maximum population size, and both are positive real constant parameters (not variables).

In the following example we consider a system of 2 first-order ODEs which includes the population of another species that is a predator of the first species. Let us call $P(t)$ the population size of the prey species and $K(t)$ the population size of the predator species, this system can be modelled as

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{M} - aPK \right), \quad \frac{dK}{dt} = aPK - dK.$$

Here, a is the predation rate, d is the death rate of the predator, and both are real positive parameters (not variables).

- 1) Compute all equilibria of the above ODE system. Hint: the equilibria might contain some or all parameters, r , M , a , d , which are all positive real numbers.

Solution. The equilibria can be found imposing $\dot{P} = 0$ and $\dot{K} = 0$. There are in total 3 equilibria of this system of ODEs. The equation $\dot{K} = aPK - dK = K(aP - d) = 0$ is satisfied for either $P = \frac{d}{a}$ or $K = 0$.

For $K = 0$ the equation $\dot{P} = rP(1 - \frac{P}{M}) = 0$ is satisfied either for $P = 0$ or for $P = M$, hence we have two equilibria at the points $(0, 0)$ and $(M, 0)$ in the (P, K) plane.

For $P = \frac{d}{a}$ the equation $\dot{P} = r \frac{d}{a} (1 - \frac{d}{aM} - a \frac{d}{a} K) = r \frac{d}{a} (1 - \frac{d}{aM} - dK) = 0$ is satisfied for $1 - \frac{d}{aM} = dK$. Thus, we have $K = \frac{1}{d} - \frac{1}{aM}$, which responds to the third equilibrium point $(\frac{d}{a}, \frac{1}{d} - \frac{1}{aM})$.

- 2) Based on the results in (1), find out the condition that the parameters need to satisfy for obtain an equilibrium point of this dynamical system describing the coexistence of the two species.

Solution. Among the three equilibria listed above, $(0, 0)$ and $(M, 0)$ refer to the case where $K = 0$, which means the predator species has died out. Only the equilibrium point $(\frac{d}{a}, \frac{1}{d} - \frac{1}{aM})$ can indicate the coexistence of the two species as long as $K > 0$, i.e. as long as the condition $\frac{1}{d} - \frac{1}{aM} > 0$ is satisfied. Thus, the two species can coexist at an equilibrium point only if $d < aM$. This means the death rate of the predator must be smaller than the product of the predation rate and the maximum population size of the prey species.

- 3) If one of the two species go extinct, i.e. $P = 0$ or $K = 0$, write down the ODE for the dynamics of the remaining species and compute the corresponding equilibria.

Solution. If the prey goes extinct, $P = 0$, then the ODE of the predator is $\frac{dK}{dt} = aPK - dK = -dK$. Thus the predator population size is governed by $\frac{dK}{dt} = -dK$. This refers to an exponential decay of the predator. The only equilibria of the predator population is $K = 0$. This means the predator will eventually go extinct if the prey goes extinct.

If the predator goes extinct, $K = 0$, then the ODE of the predator is $\frac{dP}{dt} = rP \left(1 - \frac{P}{M} - aPK\right) = rP \left(1 - \frac{P}{M}\right)$. Thus the prey population size is governed by the logistic equation $\frac{dP}{dt} = rP \left(1 - \frac{P}{M}\right)$. There are two equilibria of the prey population is $P = 0$ and $P = M$. This indicates that if the predator goes extinct, we can also have the population of the prey is not extinct.