System of 1^{st} order ODEs

Suppose to have n - dependent variables

$$y_1(x),\ y_2(x),\ y_3(x) \ \ldots \qquad y_n(x)$$

each dependent variable is a function of a SINGLE independent variable x.

A system of 1^{st} order ODEs

is a set of n 1^{st}-order differential equations whose normal form reads

$$y_1'(x) = f_1(x, y_1, y_2 \ldots \qquad y_n)$$

$$y_2'(x) = f_2(x, y_1, y_2 \ldots \qquad y_n)$$

$$\vdots$$

$$y_n'(x) = f_n(x, y_1, y_2, \ldots \ y_n)$$

In matrix form $(n=2)$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} f_1(x, y_1, y_2) \\ f_2(x, y_1, y_2) \end{pmatrix}$$

The IVP for a system of $n=2$ ODEs

ODE: $$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} f_1(x, y_1, y_2) \\ f_2(x, y_1, y_2) \end{pmatrix} \qquad (1)$$

I.C.s $$y_1(a) = b_1 \qquad y_2(a) = b_2 \qquad (2)$$

<u>Generalized Picard-Lindelöf theorem</u>

The IVP given by Eq. (1) and (2) has <u>one and only one</u> solution i a cuboid D of the form

$$|x-a| \leq A \qquad |y_1 - b_1| \leq B_1 \qquad |y_2 - b_2| \leq B_2$$

If a) both functions $f_1(x, y_1, y_2)$, $f_2(x, y_1, y_2)$ are continuous in D.

b) the partial derivatives $\frac{\partial f_1}{\partial y_1}$ $\frac{\partial f_2}{\partial y_1}$ $\frac{\partial f_1}{\partial y_2}$ $\frac{\partial f_2}{\partial y_2}$ are continuous in D

The equivalence between a single n-order ODE and a system of n 1^{st} order ODEs

Consider a single n-order ODE in

$$\boxed{y^{(n)} = F(x, y, y', y'', \ldots y^{(n-1)})} \quad (3)$$

This single n-order ODE is equivalent to a system of n 1^{st}-order ODEs

In fact we can define the functions

$y_1(x) = y(x)$, $y_2(x) = y'(x)$, $y_3(x) = y''(x)$ .. $y_n(x) = y^{(n-1)}(x)$

or

$$\begin{cases} y_1(x) = y(x) \\ y_2(x) = y'(x) \\ \vdots \\ y_n(x) = y^{(n-1)}(x) \end{cases} \quad \begin{array}{c} \text{Differentiating} \\ \text{we} \\ \text{set} \Rightarrow \end{array} \quad \begin{cases} y'_1(x) = y'(x) = y_2 \\ y'_2(x) = y''(x) = y_3 \\ \vdots \\ y'_{n-1}(x) = y^{(n-1)}(x) = y_n \\ y'_n(x) = y^{(n)}(x) = F(x, y_1, y_2, \ldots y_n) \end{cases}$$

Therefore Eq. (3) is equivalent to a system of ODEs

of the form

Therefore Eq. 13) is equivalent to a system of n 1^{st}-order ODEs

$$\begin{cases} y_1'(x) = y_2(x) \\ y_2'(x) = y_3(x) \\ \vdots \\ y_{m-1}'(x) = y_m(x) \\ y_m'(x) = F(x, y_1, y_2, \dots y_m) \end{cases}$$

System of n- ~~or~~ ODEs of 1^{st}- order.

For a second order ODE we can use the methods that

(A) we will discuss in weeks 4-6.
(methods for solving 2^{nd}- order ODEs)

(B) We will discuss in weeks 8-12
(methods for solving ~~2 1st~~ a system of 2 1^{st} order ODEs).

<u>Example</u> Reduce the following 2^{nd} order ODE to a system of 2 1^{st}- order ODEs.

$y'' = 6y - 4y'$ $\qquad$ $y'' = F(x, y, y') = 6y - 4y'$

$$\begin{cases} y_1(x) = y(x) \\ y_2(x) = y'(x) \end{cases}$$ We define $y_1(x)$ and $y_2(x)$

$$\begin{cases} y_1'(x) = y'(x) = y_2 \\ y_2'(x) = y''(x) = 6\underset{y_1}{\underset{\parallel}{y}} - 4\underset{y_2}{\underset{\parallel}{y'}} = 6y_1 - 4y_2 \end{cases}$$

System of 2 1st- order ODEs

$$\begin{cases} y_1'(x) = y_2(x) \\ y_2'(x) = 6y_1 - 4y_2 \end{cases}$$

This system of n 1st- order ODEs is expressed in terms of the independent variable x but it can be also transformed into a system of n 1st- order ODEs with independent variable t upon identifying $x \equiv t$

With this transformation

$y_1(x) = y_1(t)$ $\quad$ $y_2(x) = y_2(t)$ .. $\quad$ $y_n(x) = y_n(t)$

Additionally we have

$$\frac{d y_1(x)}{dx} \longrightarrow \frac{d y_1(t)}{dt} = \dot{y}_1$$

$$\frac{d y_2(x)}{dx} \longrightarrow \frac{d y_2(t)}{dt} = \dot{y}_2$$

$$\vdots$$

$$\frac{d y_n(x)}{dx} \longrightarrow \frac{d y_n(t)}{dt} = \dot{y}_n = F(t, y_1, y_2, \ldots y_n)$$

The n-order ODE

$$y^{(n)} = F(x, y, y', \ldots y^{(n-1)})$$

is equivalent to the system

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = y_3 \\ \vdots \\ \dot{y}_{n-1} = y_n \\ \dot{y}_n = F(t, y_1, y_2, \ldots y_n) \end{cases}$$

Example $\qquad y'' = 6y - 4y'$

We have shown that this ODE is equivalent to

$$\begin{cases} y_1'(x) = y_2 \\ y_2'(x) = 6y_1 - 4y_2 \end{cases}$$

By putting $x \equiv t$ we derive the equivalente to

$$\begin{cases} \dot{y}_1(t) = y_2 \\ \dot{y}_2(t) = 6y_1 - 4y_2 \end{cases}$$

$\square$