Exploratory problem

Solve the motion of a mass attached to the ceiling by a spring (no viscosity)

Newton's 2^{nd} law

$$m\ddot{y} = f_g + f_{el}$$

$$f_g = mg$$

$$m, g, k, \ell \in \mathbb{R}^+$$

$$f_{el} = -k(y-\ell)$$

$$m\ddot{y} = mg - k(y-\ell) \implies m\ddot{y} + ky = mg + k\ell$$

$$\boxed{\ddot{y} + \frac{k}{m}y = g + \frac{k}{m}\ell}$$

2^{nd}-order
linear
inhomogeneous

independent variable? t

dependent variable y $\quad y = y(t)$

1) Solve the homogeneous problem

$$\ddot{y} + \frac{k}{m}y = 0$$

Characteristic equation

$$\lambda^2 + \frac{k}{m} = 0 \qquad \lambda = \pm\sqrt{-\frac{k}{m}}$$

putting $\frac{k}{m} = \omega^2$ $\qquad \lambda_{1,2} = \pm i\omega$

$\omega = \sqrt{\frac{k}{m}}$ arch

Case $\lambda_1 = \alpha + i\beta$ $\quad \lambda_2 = \alpha - i\beta$ $\qquad \alpha = 0 \quad \beta = \omega$

① Solve the homogeneous problem

$$\ddot{y} + \omega^2 y = 0 \qquad \text{where} \qquad \omega^2 = \frac{k}{m}$$

Characteristic equation

$$\lambda^2 + \omega^2 = 0, \; \lambda^2 = -\omega^2 \Rightarrow \; \lambda = \pm i\omega$$

$$\lambda_1 = \alpha + i\beta \qquad \text{where} \qquad \alpha = 0$$
$$\lambda_2 = \alpha - i\beta \qquad \qquad \beta = \omega$$

General solution

$$y_h(t) = e^{\alpha t}\left(A \cos \beta t + B \sin \beta t\right)$$

$$y_h(t) = \left(A \cos \omega t + B \sin \omega \cdot t\right)$$

$$\boxed{y_h(t) = A \cos \omega t + B \sin \omega t}$$

② Find the particular solution.

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = f(t)$$

$$\ddot{y} + \omega^2 y = \underbrace{g + \frac{k}{m} \ell}_{\text{''} f(t) = p(t) e^{at}} \qquad \text{where } \omega^2 = \frac{k}{m}$$

with $a \neq \lambda_1, \lambda_2 \neq a$

$$a = 0 \qquad f(t) = g + \frac{k}{m}\ell \Rightarrow$$ I can apply the educated guess method!

According to the educated guess method we look for solutions

$$y_p(t) = Q(t)\, e^{at} = d_0$$

$$\uparrow$$

$a = 0$

$Q(t)$ · polynomial of degree $k=0$

We calculate $\dot{y}_p(t) = 0$, $\ddot{y}(t) = 0$

We insert $y_p(t)$, $\dot{y}_p(t)$, $\ddot{y}_p(t)$ into the ODE

$$\ddot{y} + \frac{k}{m} y = g + \frac{k}{m} \ell$$

$$\frac{k}{m} d_0 = g + \frac{k}{m} \ell \Rightarrow d_0 = \frac{m}{k} g + \ell$$

$$\boxed{y_p(t) = \frac{m}{k} g + \ell}$$

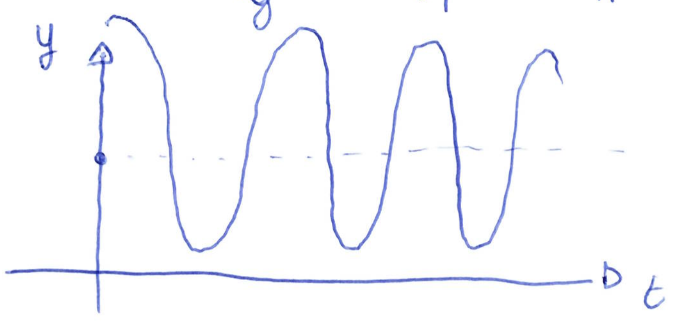
(4) The general solution to the inhomogeneous problem

$$\ddot{y} + \frac{k}{m} y = g + \frac{k}{m} \ell$$

is

$$y_g(t) = y_p(t) + y_h(t) = \frac{m}{k} g + \ell + A \cos\omega t + B \sin\omega t$$

where $A, B \in \mathbb{R}$



Summary of considered ODEs

Weeks 1-2   5 types of 1^{st}-order ODEs

1. $y' = f(x)$ (calculus)

2. $y' = f(x)\,g(y)$ Separable ODE

3. $$\left.\begin{aligned} &y' = f(ax+by+c) \\ &y' = f\left(\frac{y}{x}\right) \end{aligned}\right\} \text{ Reducible to separable ODEs.}$$

4. Linear 1^{st}-order ODEs

$$y' = A(x)y + B(x)$$

a) Homogeneous $\quad B(x) = 0$

$$y' = A(x)y \quad \text{- separable}$$

b) Inhomogeneous $\quad B(x) \neq 0$

$$y' = A(x)y + B(x)$$

Variation of parameter method.

5. Exact 1^{st}-order ODEs $\quad P(x,y) + Q(x,y)\dfrac{dy}{dx} = 0$

if and only if $\quad \dfrac{\partial}{\partial y} P(x,y) = \dfrac{\partial}{\partial x} Q(x,y)$

Week ~~3~~ 4 – 5 2^{nd} - order ODEs

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = f(x)$$

① Homogeneous ($f(x)=0$) with constant coefficients

$$a_2 y'' + a_1 y' + a_0 y = 0$$

$\Rightarrow$ Characteristic equation $\Rightarrow$ General solution

② Euler ODEs – Homogeneous.

$$a x^2 y'' + b x y' + c y = 0$$

Reducible to ODEs with constant coefficient

$z = y(x(t))$ where $x = e^t$ $x > 0$

$$a\ddot{z} + (b-a)\dot{z} + cz = 0$$

③ Inhomogeneous with constant coefficients

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

a) Variation of parameter method

b) Educated guess method

Only applicable for $f(x) = p(x)e^{ax}$

or $f(x) = p(x)\begin{cases}\cos ax \\ \sin ax\end{cases}$

$a \neq \lambda_1 \quad a \neq \lambda_2$