Consider the dynamical system

$$\begin{cases} \dot{y}_1 = y_1 + e^{y_2} - \cos y_2 & = f_1(y_1, y_2) \\ \dot{y}_2 = 3y_1 - y_2 - \sin y_2 & = f_2(y_1, y_2) \end{cases} \qquad (1)$$

linearise the system around the equilibrium point $(y_1, y_2) = (0,0)$ and establish the nature of the phase portrait of the linerised system.

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① Let check that $y_1 = y_2 = 0$ is an equilibrium point

$(0,0)$ is an equilibrium point iff $f_1(0,0) = f_2(0,0) = 0$

$$f_1(0,0) = 0 + \underbrace{e^0}_{=1} - \underbrace{\cos 0}_{=1} = 0 \quad \checkmark$$

$$f_2(0,0) = 0 - 0 - \underbrace{\sin 0}_{=0} = 0 \quad \checkmark$$

$(0,0)$ is an equilibrium point.

② The linearised dynamical system reads

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} \left.\frac{\partial f_1}{\partial y_1}\right|_{0,0} & \left.\frac{\partial f_1}{\partial y_2}\right|_{0,0} \\ \left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} & \left.\frac{\partial f_2}{\partial y_2}\right|_{0,0} \end{pmatrix}$$

$$f_1(y_1,y_2) = y_1 + e^{y_2} - \cos y_2$$

$$f_2(y_1,y_2) = 3y_1 - y_2 - \sin y_2$$

$$\left.\frac{\partial f_1}{\partial y_1}\right|_{(0,0)} = 1 \quad , \quad \left.\frac{\partial f_1}{\partial y_2}\right|_{(0,0)} = \left. e^{y_2} + \sin y_2 \right|_{(0,0)} = e^0 + \sin 0 = 1$$

$$\left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} = 3 \quad , \quad \left.\frac{\partial f_2}{\partial y_2}\right|_{(0,0)} = \left. -1 - \cos y_2 \right|_{(0,0)} = -2$$

Therefore

$$A = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \quad \square$$

~~b)~~ The linearised system is

$$\dot{Y} = AY \quad \text{with } Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$$

$\square$

Let us calculate the eigenvalues of A.

$$\det(A - \lambda Id) = \det\begin{pmatrix} 1-\lambda & 1 \\ 3 & -2-\lambda \end{pmatrix} = -(1-\lambda)(2+\lambda) - 3 = 0$$

$$(1-\lambda)(-2-\lambda) = (1-\lambda)(-1)(2+\lambda)$$

$\lambda^2 + \lambda - 5 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1+20}}{2} = \begin{cases} \dfrac{-1+\sqrt{21}}{2} = \lambda_1 \\[2ex] \dfrac{-1-\sqrt{21}}{2} = \lambda_2 \end{cases}$$

$\boxed{\lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 \neq \lambda_2, \quad \lambda_1 > 0 \quad \lambda_2 < 0}$ SADDLE

Consider the dynamical system

$$\dot{y}_1 = -2y_1 - 3y_2 + y_1^5 = f_1(y_1, y_2)$$

$$\dot{y}_2 = y_1 + y_2 - y_2^2 = f_2(y_1, y_2)$$

linearise the dynamical system around the equilibrium point $(0,0)$ and establish the nature of the phase portrait of the linearised system.

① Check that $(y_1, y_2) = (0,0)$ is an equilibrium point.

$f_1(0,0) = 0$ ✓

$f_2(0,0) = 0$ ✓

$(0,0)$ is an equilibrium point.

(2) The linearised dynamical system is

$$\dot{Y} = AY \quad \text{with } Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

and

$$A = \begin{pmatrix} \left.\frac{\partial f_1}{\partial y_1}\right|_{(0,0)} & \left.\frac{\partial f_1}{\partial y_2}\right|_{(0,0)} \\ \left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} & \left.\frac{\partial f_2}{\partial y_2}\right|_{(0,0} \end{pmatrix}$$

$$f_1(y_1; y_2) = -2y_1 - 3y_2 + y_1^5$$

$$f_2(y_1, y_2) = y_1 + y_2 - y_2^2$$

$$\left.\frac{\partial f_1}{\partial y_1}\right|_{(0,0)} = -2 + 5y_1^4\Big|_{(0,0)} = -2, \quad \left.\frac{\partial f_1}{\partial y_2}\right|_{(0,0)} = -3$$

$$\left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} = 1, \quad \left.\frac{\partial f_2}{\partial y_2}\right|_{(0,0)} = 1 - 2y_2\Big|_{(0,0)} = 1$$

$$A = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$$

Which is the phase portrait?

$$\det(A - \lambda Id) = \det\begin{pmatrix} -2-\lambda & -3 \\ 1 & 1-\lambda \end{pmatrix} = -(2+\lambda)(1-\lambda) + 3 = 0$$

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \begin{cases} \dfrac{-1+\sqrt{3}i}{2} = \lambda_1 \\[2ex] \dfrac{-1-\sqrt{3}i}{2} = \lambda_2 \end{cases}$$

$\lambda_1 \neq \lambda_2$ and $\lambda_1 = \alpha + i\beta$ $\quad \alpha, \beta \in \mathbb{R} \quad \beta > 0$

$\lambda_2 = \alpha - i\beta$ $\quad \lambda_1, \lambda_2$ complex conjugate.

$\boxed{\alpha = -\frac{1}{2} < 0}$ $\qquad \beta = \frac{\sqrt{3}}{2}$

The phase portrait is a <u>spiral in</u>

(0,0) is a <u>stable focus</u>

Week 8 Formative Assessment Ex. (c)

Graph the parametric curve given by

$$\begin{cases} y_1(t) = e^{-t} \sin 3t \\ y_2(t) = e^{-t} \cos 3t \end{cases} \quad \text{for } 0 \leq t \leq \infty$$

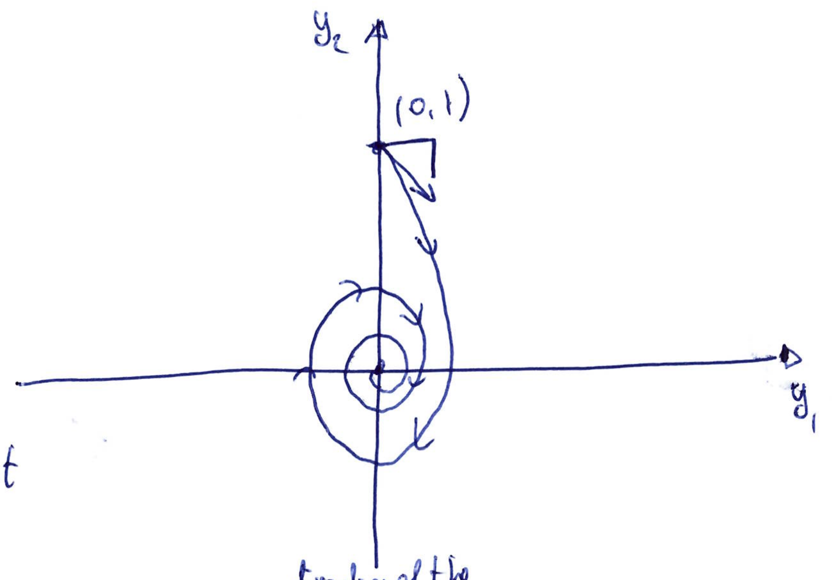
$$\rho^2 = y_1^2 + y_2^2 = e^{-2t} \sin^2 3t + e^{-2t} \cos^2 3t = e^{-2t} \underbrace{(\sin^2 3t + \cos^2 3t)}_{1}$$

$$\rho^2 = e^{-2t} \Rightarrow \rho = e^{-t}$$

$\rho(0) = e^0 = 1$ As $t \to \infty$ $\rho(t) \to 0$

This is a <u>spiral in</u>

$$\begin{cases} y_1(0) = e^0 \sin 0 = 0 \\ y_2(0) = e^0 \cos 0 = 1 \end{cases}$$



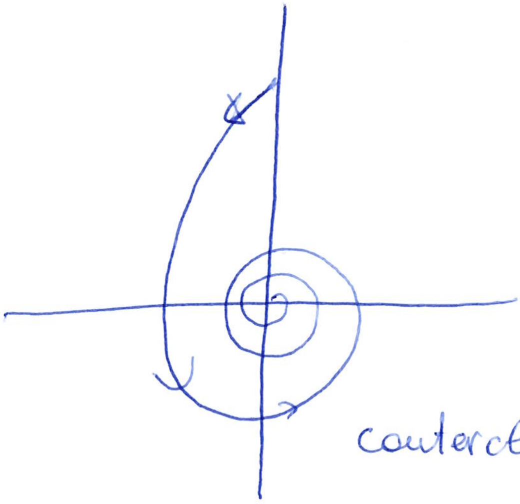
$$\begin{cases} \dot{y}_1(t) = -e^{-t} \sin 3t + 3e^{-t} \cos 3t \\ \dot{y}_2(t) = -e^{-t} \cos 3t - 3e^{-t} \sin 3t \end{cases}$$

$$\begin{cases} \dot{y}_1(0) = \underline{3} \\ \dot{y}_2(0) = -1 \end{cases}$$

The ~~d~~ direction of the spiral is clockwise.

$\dot{y}_1(0) < 0$

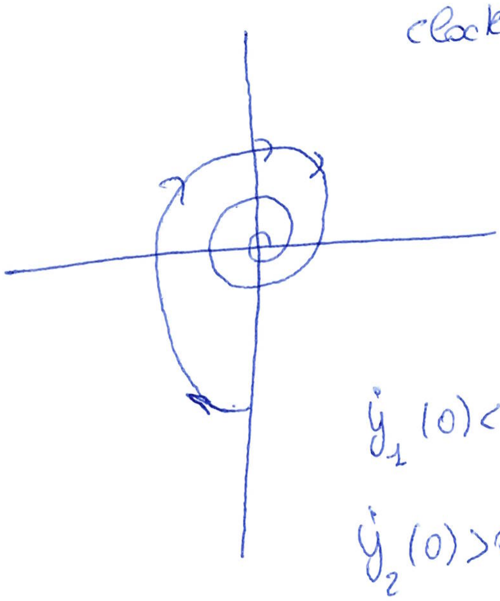
$\dot{y}_2(0) < 0$



couterclock wise

If the initial condition ~~is~~ is $(y_1(0), y_2(0)) = (0, -1)$

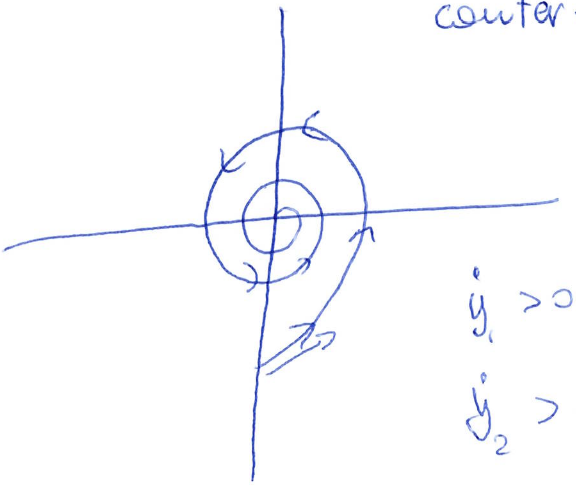
clockwise



$\dot{y}_1(0) < 0$

$\dot{y}_2(0) > 0$

couter-clockwise



$\dot{y}_1 > 0$

$\dot{y}_2 > 0$