Consider the system of ODEs

$$\dot{Y} = AY \quad (1) \quad \text{where} \quad A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$$

What type of fixed point is $(0,0)^T$?

Is the zero solution $Y(t) = (0,0)^T$ stable?

Is it asymptotically stable?

What kind of trajectories do we see in the phase portrait?

<u>Solution</u> Let us determine the eigenvalues of A.

$$\det(A - \lambda Id) = \det\begin{pmatrix} -\lambda & -4 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 4 = 0$$

Therefore $\lambda^2 = -4$ $\quad \begin{cases} \lambda_1 = 2i \\ \lambda_2 = -2i \end{cases}$ $\quad \begin{matrix} \lambda_1 = \alpha + i\beta \\ \lambda_2 = \alpha - i\beta \end{matrix}$

where $\boxed{\alpha = \max(Re\lambda_1, Re\lambda_2) = 0}$, $\beta = 2$,

- The fixed point is a CENTRE ✓
- The zero solution is STABLE ✓
- The zero solution is <u>NOT</u> asymptotically stable ✓

In order to determine whether the trajectory is a circle or an ellipse we need to solve the IVP with IC.

$$Y(0) = \begin{pmatrix} a \\ b \end{pmatrix}$$

① Determine the general solution.

$$Y(t) = D_1 e^{2it} u_1 + D_2 e^{-2it} u_2 \qquad (2)$$

If we impose $Y(t)$ to be real $\forall t$ then

u_1, u_2 are complex conjugate

D_1, D_2 are complex conjugate

so that $c_1 = D_1 e^{2it} u_1$ is comple conjugate of

$c_2 = D_2 e^{-2it} u_2$.

In order to calculate u_1 we need to solve

$$A u_1 = \lambda_1 u_1 \qquad \text{where} \quad u_1 = \begin{pmatrix} P_1 \\ q_1 \end{pmatrix}, \ \lambda_1 = 2i$$

$$\begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_1 \\ q_1 \end{pmatrix} = 2i \begin{pmatrix} P_1 \\ q_1 \end{pmatrix} \Rightarrow \begin{cases} -4q_1 = 2i P_1 \\ = 2i q_1 = P_1 \end{cases}$$

$P_1 = 2 \qquad q_1 = -i \qquad \boxed{u_1 = \begin{pmatrix} 2 \\ -i \end{pmatrix}} \qquad 2i(-i) = 2$

Therefore $\quad u_1 = \begin{pmatrix} 2 \\ -i \end{pmatrix} \qquad u_2 = \begin{pmatrix} 2 \\ i \end{pmatrix}$

The general solution is given by (2)

$$Y(t) = D_1 e^{2it} u_1 + D_2 e^{-2it} u_2$$

Since $c_1 = D_1 e^{2it} u_1$ and $c_2 = D_2 e^{-2it} u_2$ are complex conjugate they satisfy.

$$c_1 + c_2 = 2 \operatorname{Re} c_1$$

therefore $\quad \boxed{Y(t) = 2 \operatorname{Re}\left[D_1 e^{2it} u_1\right]}$

(2) We impose the I.C. $\quad Y(0) = \begin{pmatrix} a \\ b \end{pmatrix}$

$$Y(0) = 2 \operatorname{Re}\left[D_1 \begin{pmatrix} 2 \\ -i \end{pmatrix}\right] = \begin{pmatrix} 4 \operatorname{Re} D_1 \\ 2 \operatorname{Im} D_1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\operatorname{Re}\left(D_1 (-i)\right) = \operatorname{Re}\left[(\operatorname{Re} D_1 + \underbrace{i \operatorname{Im} D_1)(-i)}_{1}\right] = \operatorname{Im} D_1$$

Therefore $\quad \left.\begin{aligned} 4 \operatorname{Re} D_1 &= a \\ 2 \operatorname{Im} D_1 &= b \end{aligned}\right\} \left\{\begin{aligned} \operatorname{Re} D_1 &= \frac{a}{4} \\ \operatorname{Im} D_1 &= \frac{b}{2} \end{aligned}\right. \Rightarrow \boxed{D_1 = \frac{a}{4} + i \frac{b}{2}}$

$$Y(t) = 2\,\mathrm{Re}\left[D_1 e^{2it} u_1\right] = 2\,\mathrm{Re}\left[\left(\frac{a}{4} + i\frac{b}{2}\right) e^{2it} \begin{pmatrix} 2 \\ -i \end{pmatrix}\right] \quad *$$

We want to show that this expression is equivalent to

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} a\cos 2t - 2b\sin 2t \\ \frac{a}{2}\sin 2t + b\cos 2t \end{pmatrix} \begin{matrix} \checkmark \\ \checkmark \end{matrix} \quad **$$

Starting with *

$$Y(t) = 2\,\mathrm{Re}\left[D_1 e^{2it} \begin{pmatrix} 2 \\ -i \end{pmatrix}\right] = \begin{pmatrix} 4\,\mathrm{Re}\left(D_1 e^{2it}\right) \\ 2\,\mathrm{Im}\left(D_1 e^{2it}\right) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$4\,\mathrm{Re}\left(D_1 e^{2it}\right) = 4\,\mathrm{Re}\left[\left(\frac{a}{4} + i\frac{b}{2}\right)\left(\cos 2t + i\sin 2t\right)\right] = y_1(t)$$

If $\bar{c}_1, \bar{c}_2$ are complex numbers

$$\mathrm{Re}\,\bar{c}_1\bar{c}_2 = \mathrm{Re}\,\bar{c}_1\,\mathrm{Re}\,\bar{c}_2 - \mathrm{Im}\,\bar{c}_1\,\mathrm{Im}\,\bar{c}_2$$

$$y_1(t) = 4\left[\frac{a}{4}\cos 2t - \frac{b}{2}\sin 2t\right] = a\cos 2t - 2b\sin 2t \quad \checkmark$$

$$y_2(t) = 2\,\mathrm{Im}\left(D_1 e^{2it}\right) = 2\,\mathrm{Im}\left[\left(\frac{a}{4} + i\frac{b}{2}\right)\left(\cos 2t + i\sin 2t\right)\right]$$

If $\bar{c}_1, \bar{c}_2$ are complex numbers

$$\mathrm{Im}\left(\bar{c}_1 \bar{c}_2\right) = \mathrm{Re}\,\bar{c}_1\,\mathrm{Im}\,\bar{c}_2 + \mathrm{Im}\,\bar{c}_1\,\mathrm{Re}\,\bar{c}_2$$

$$y_2(t) = 2\left[\frac{a}{4}\sin 2t + \frac{b}{2}\cos 2t\right] = \frac{a}{2}\sin 2t + b\cos 2t \quad \checkmark$$

$\square$

Therefore we have shown that

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} a\cos 2t - 2b\sin 2t \\ \frac{a}{2}\sin 2t + b\cos 2t \end{pmatrix}$$

(3) Let us characterise the trajectory and establish if it is a circle or an ellipse.

$$y_1^2 + 4y_2^2 = \left(a\cos 2t - 2b\sin 2t\right)^2 + 4\left(\frac{a}{2}\sin 2t + b\cos 2t\right)^2 =$$

$$= a^2\cos^2 2t + 4b^2\sin^2 2t - 4ab\cos 2t\sin 2t +$$

$$4\left(\frac{a^2}{4}\sin^2 2t + b^2\cos^2 t + ab\sin 2t\cos 2t\right)$$

$$= a^2\cos^2 2t + 4b^2\sin^2 2t - \cancel{4ab\cos 2t\sin 2t}$$

$$a^2\sin^2 2t + 4b^2\cos^2 2t + \cancel{4ab\cos 2t\sin 2t}$$

$$= a^2\underbrace{\left(\cos^2 2t + \sin^2 2t\right)}_{=1} + 4b^2\underbrace{\left(\sin^2 2t + \cos^2 2t\right)}_{=1} = a^2 + 4b^2$$

$$\boxed{y_1^2 + 4y_2^2 = a^2 + 4b^2}$$

The trajectory is an ellipse