Mock Quiz Week 11

Which of the following functions $V(y_1, y_2)$ is a Lyapunov function for the dynamical system

$$\begin{cases} \dot{y}_1 = -y_1 y_2^2 = f_1(y_1, y_2) \\ \dot{y}_2 = -2y_1^2 y_2 - 3y_2 = f_2(y_1, y_2) \end{cases}$$

① Find the fixed points

$$\begin{cases} 0 = \dot{y}_1 = -y_1 y_2^2 \\ 0 = \dot{y}_2 = -2y_1^2 y_2 - 3y_2 = y_2(-2y_1^2 - 3) \end{cases} \qquad \begin{cases} y_1 = 0 \text{ or } y_2 = 0 \\ y_2 = 0 \end{cases}$$

$$\boxed{y_2 = 0}$$

The fixed point $(a, 0)$ $\forall a \in \mathbb{R}$

We need to check that there exist a fixed point $(a^*, 0)$

a) $V(a^*, 0) = 0$

b) $V(y_1, y_2) > 0 \qquad \forall (y_1, y_2) \neq (a^*, 0)$

c) $D_f(V) < 0 \qquad \forall (y_1, y_2) \neq (a^*, 0)$

or

$D_f(V) \leq 0 \qquad \forall (y_1, y_2) \neq (a^*, 0)$

(1) $V(y_1, y_2) = -y_1 y_2^2$

a) $V(a^*, 0) = -y_1 y_2^2 \Big|_{(y_1,y_2)=(a^*,0)} = 0$ ✓

b) $V(y_1 y_2) = -y_1 y_2^2 > 0 \quad \forall (y_1, y_2) \neq (a^*, 0)$

$y_1 = 1 \quad y_2 = 1 \quad V(y_1, y_2) = -1$

$y_1 > 0 \quad y_2 \neq 0 \quad V(y_1 y_2) < 0$

X

This is NOT a valid Lyapunov function.

(2) $V(y_1 y_2) = y_2^3$

a) $V(a^*, 0) = 0$ ✓

b) $V(y_1, y_2) = y_2^3 > 0$ for all $(y_1, y_2) \neq (a^*, 0)$

If $y_2 < 0 \quad V(y_1, y_2) < 0$ X

This is NOT a valid Lyapunov function.

(3) $V(y_1, y_2) = \sin(y_1^2 + y_2^2)$

a) $V(y_1, y_2) = \sin(y_1^2 + y_2^2) = 0$ $\qquad (y_1, y_2) = (a^*, 0)$

$\sin 0 = 0$ $\qquad a^* = 0$ ✓

$a^* = \pi n$

b) $V(y_1, y_2) > 0$ $\qquad \forall (y_1, y_2) \neq (\pi n, 0)$

$\sin(y_1^2 + y_2^2) > 0$

$y_1^2 + y_2^2 = \frac{3\pi}{2}$ $\qquad \sin(y_1^2 + y_2^2) = \sin\left(\frac{3\pi}{2}\right) = -1$

X This is NOT a valid Lyapunov function.

(4) $V(y_1, y_2) = \sinh(y_1^2 + y_2^2)$

a) $V(y_1, y_2)\Big|_{(y_1, y_2) = (a^*, 0)} = \sinh(y_1^2 + y_2^2)\Big|_{(y_1, y_2) = (0, 0)} = 0$ ✓

$a^* = 0$

b) $V(y_1, y_2) > 0$ $\qquad \forall (y_1, y_2) \neq (0, 0)$

$\sinh(y_1^2 + y_2^2) > 0$ $\qquad y_1^2 + y_2^2 > 0$ $\qquad (y_1, y_2) \neq (0, 0)$

✓

c) $D_f(V) = \dfrac{\partial V}{\partial y_1}\dot{y}_1 + \dfrac{\partial V}{\partial y_2}\dot{y}_2$

$V(y_1, y_2) = \sinh(y_1^2 + y_2^2)$

$\dfrac{\partial V}{\partial y_1} = \cosh(y_1^2 + y_2^2)\, 2y_1$

$\dfrac{\partial V}{\partial y_2} = \cosh(y_1^2 + y_2^2)\, 2y_2$

$\dot{y}_1 = -y_1 y_2^2$

$\dot{y}_2 = -2y_1^2 y_2 - 3y_2$

$$D_f(V) = \cosh(y_1^2+y_2^2)\, 2y_1 \underbrace{\left(-y_1 y_2^2\right)}_{\dot{y}_1} + \cosh(y_1^2+y_2^2)\, 2y_2 \underbrace{\left(-2y_1^2 y_2 - 3y_2\right)}_{\dot{y}_2}$$

$$= \underbrace{\cosh(y_1^2+y_2^2)}_{>0}\left[-2y_1^2y_2^2 - 4y_1^2y_2^2 - 6y_2^2\right] \le 0$$

(0,0) is Lyapunov stable.

This a valid Lyapunov function.