2019 Final Marks Scheme

① (a) $m \frac{dv}{dt} = mg - \gamma v$ with

[5]

$m = 10$, $g = 9.8$, $\gamma = 2$ becomes

$10 \frac{dv}{dt} = 10(9.8) - 2v$, or equivalently

$\Rightarrow \frac{dv}{dt} = 9.8 - \frac{1}{5}v = \frac{49 - v}{5}$. This

+1 plug in

Equation is separable: +1

$$\int \frac{\frac{dv}{dt}}{v - 49}\, dt = \int -\frac{1}{5}\, dt$$

+1 for any attempt

+3

$\Rightarrow \ln|v - 49| = -\frac{1}{5}t + C$ (+1, +1)

+1 $\Rightarrow$ So $v(t) = 49 + Ke^{-t/5}$ is the general solution.

① (b) Substituting the IC

[5]

+3 $v(0) = 49 = 49 + K \cdot e^0$

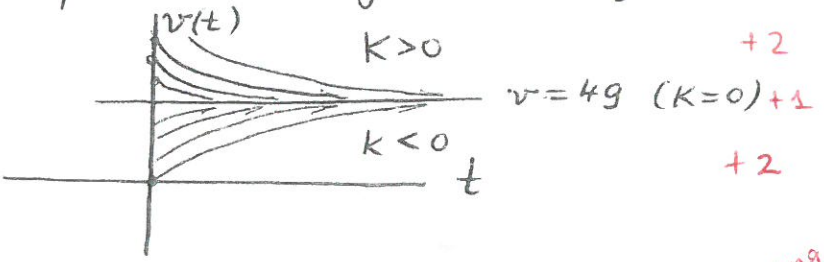
$\Rightarrow K = 0$

+2 gives the specific solution $v(t) = 49$, the constant function.

no punishment if v(t) incorrect but procedure correct

① (c) $v(t) = 49 + Ke^{-t/5}$: note that K can be positive, negative or zero

[5]



0 pts if totally wrong graph / wrong axes etc.

~~+1 for attempt~~ +2 for partially correct attempt

① ⓓ points on vertical axis are initial velocities for the falling object

5

+1 (e.g. origin denotes mass is being dropped from Rest)

+3 since $\lim_{t \to \infty} (49 + Ke^{-t/5}) = 49$ for any $K \in \mathbb{R}$, we see that irrespective of the initial velocity of the falling object, it's velocity will approach (or already equal)

+1 49 m/s (terminal velocity).

Q1 a, b, c coursework
d unseen

② ⓐ $\begin{cases} y' = y\tan x + \sin x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ y(0) = 1 \end{cases}$

$\boxed{10}$

First solve the homogeneous problem

+4

+2 for attempt

$$y' = y\tan x \implies \frac{y'}{y} = \tan x$$

$$\ln|y| = \int \tan x\, dx$$

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx \underset{u=\cos x}{=} -\ln|\cos x| + C$$

So $|y| = e^{-\ln|\cos x| + C} = e^C|\sec x|$

$\implies y = K\sec x$.

Now use variation of parameters for the inhomogeneous problem: $y(x) = K(x)\sec x$

$$\implies y'(x) = K'(x)\sec x + K(x)\sec x\tan x$$

+4

$y' = y\tan x + \sin x$ becomes

$$K'\sec x + \cancel{K\sec x\tan x} = \cancel{K\sec x\tan x} + \sin x$$

can also use Educated guess for these 4 pts. $y = y_h + y_p$

$\implies K'(x) = \sin x\cos x$

+2 for attempt

$\implies K(x) = \frac{1}{2}\sin^2 x + C$

Soln becomes $y(x) = \left(\frac{1}{2}\sin^2 x + C\right)\sec x$

+2

$y(0) = (0 + C)1 = 1 \implies C = 1$

$$\implies y(x) = \left(\frac{1}{2}\sin^2 x + 1\right)\sec x.$$

② ⓑ To apply Picard-Lindelöf Theorem,

$\boxed{10}$ we need to check the 3 conditions on

$\mathcal{D} = \{(x,y) : |x-0| \leq A \,,\, |y-1| \leq B\}$ +2

for $f(x,y) = y\tan x + \sin x$:

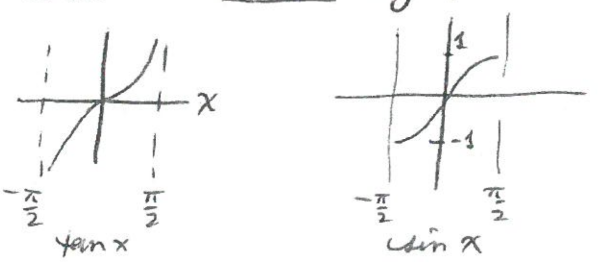
(i) f is continuous on $\mathcal{D}$ since y, $\tan x$ and $\sin x$ are continuous, NOTING that we require $-\frac{\pi}{2} < x < \frac{\pi}{2}$ so $\underline{A < \frac{\pi}{2}}$. +2

(ii) We need $A \leq B/M$, so we must find

$M = \max_{\mathcal{D}} |f(x,y)|$ +2

$= \max_{\mathcal{D}} |y\tan x + \sin x| = (1+B)\tan A + \sin A$

- for $|x| \leq A < \frac{\pi}{2}$, $\tan x$ and $\sin x$ are increasing functions



max will be at right endpoint of the interval $[-A, A]$

- y is an increasing function, so max will be at right endpoint $-B \leq y-1 \leq B$
  $\Rightarrow 1-B \leq y \leq 1+B$

(iii) $\left|\frac{\partial f}{\partial y}\right| = \left|\frac{\partial}{\partial y}(y\tan x + \sin x)\right| = |\tan x|$ is bounded on $\mathcal{D}$ since $|x| \leq A < \frac{\pi}{2}$. +1 +1 +2

② ⓑ (cont'd) Since all 3 hypotheses of the theorem are satisfied, Picard-Lindelöf guarantees a unique solution to the IVP

$$\begin{cases} y'(x) = y\tan x + \sin x \\ y(0) = 1 \end{cases}$$

+2

On the rectangle $\mathcal{D} = \{(x,y) : |x| \le A < \frac{\pi}{2},$
$|y-1| \le B\}$,

where A, B satisfy

$$A \le B/((1+B)\tan A + \sin A).$$

whatever value obtained for (ii)
incorrect M not deducted additionally

Q2 a modified from coursework
b unseen, Bookwork

10

3 a $\underbrace{\ln|xy|}_{Q(x,y)} \frac{dy}{dx} + \underbrace{x^2 + \frac{f(y)}{xy}}_{P(x,y)} = 0$

Equation is Exact if $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$ +3

$$\frac{\partial P}{\partial y} = \frac{f'(y)}{xy} + \frac{f(y)}{x}\left(\frac{-1}{y^2}\right) = \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{f(y)}{y}\right)$$ +2

$$\frac{\partial Q}{\partial x} = \frac{1}{xy}(y) = \frac{1}{x}$$ +2

so $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ becomes $\frac{1}{x}\frac{\partial}{\partial y}\left(\underbrace{\frac{f(y)}{y}}_{\text{function of } y \text{ alone}}\right) = \frac{1}{x}$

$$\Longrightarrow \left(\frac{f(y)}{y}\right)' = 1$$

so $\frac{f(y)}{y} = y + C$ which means

$$f(y) = y^2 + Cy$$ +3

10

3 b If we require $f(1) = 1$, then $C = 0$ +2
so $f(y) = y^2$. To solve $(1^2 + C \cdot 1 = 1)$ no matter what f from part a
the exact differential Equation,
we look for a solution in implicit
form $F(x,y) = C$ which satisfies

③ ⓑ (cont'd)

$$\frac{\partial F}{\partial x} \overset{(1)}{=} P(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} \overset{(2)}{=} Q(x,y).$$

$$(1) \Longrightarrow \quad F(x,y) = \int P(x,y)\,dx \quad + g(y)$$

$$= \int x^2 + \frac{f(y)}{xy}\,dx \quad + g(y)$$

$$\downarrow f(y) = y^2$$

$$= \frac{1}{3}x^3 + y\ln|x| \quad + g(y)$$

+3

$$\Rightarrow \quad \frac{\partial F}{\partial y} = \ln|x| + g'(y) \overset{(2)}{=} Q(x,y)$$

$$= \ln|xy|$$

$$= \ln|x| + \ln|y|$$

+3

$$\Rightarrow \quad g'(y) = \ln|y|$$

$$\text{so} \quad g(y) = y\ln y - y$$

$\downarrow$ say $y>0$

or use abs value in $\ln|y|$

Solution to original ODE is thus

$$\frac{1}{3}x^3 + y\ln|x| + y\ln|y| - y = C.$$

+2

Q3 Coursework

④ ⓐ $x^2 y'' - 2y = 0$

5

Let $x = e^t$ and $z(t) = y(e^t)$. Then

$\dot{z} = y'(e^t)e^t = y'(x)\cdot x$ (Chain Rule)

$\ddot{z} = y''(e^t)\, e^{2t} + y'(e^t)\, e^t$

$= y''(x)\cdot x^2 + y'(x)\cdot x$

+2

$\Rightarrow x^2 y'' = \ddot{z} - \dot{z}$ where prime denotes $\frac{d}{dx}$ and dot denotes $\frac{d}{dt}$ on the left.

Thus

$x^2 y'' - 2y = \ddot{z} - \dot{z} - 2z = 0.$

Solving $\ddot{z} - \dot{z} - 2z = 0$ using $z(t) = e^{\lambda t}$ gives characteristic equation

$\lambda^2 - \lambda - 2 = 0$

+2

$(\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 2$

So the general solution to the z-equation is $z(t) = C_1 e^{-t} + C_2 e^{2t}$ which then gives the general solution to the original equation as ($x = e^t$)

+1

$y(x) = \dfrac{C_1}{x} + C_2 x^2.$

④ ⓑ Let $\mathcal{L}(y) = x^2 y'' - 2y$. The left-end

5

and Right-end IVPs are given by Examining the BCs:

$$y(1)=0 \quad , \quad y(2)+2y'(2)=0$$

$$\Rightarrow \quad [x_1, x_2] = [1, 2] \quad , \quad \begin{matrix} \alpha = 0 & \gamma = 2 \\ \beta = 1 & \delta = 1 \end{matrix}$$

Left-End IVP

$$+2 \quad \begin{cases} \mathcal{L}(y) = 0 \\ y(1) = 0 \\ y'(1) = -1 \end{cases}$$

Right-End IVP +1

$$\begin{cases} \mathcal{L}(y) = 0 \\ y(2) = 2 \\ y'(2) = -1 \end{cases} \quad +2$$

Note: From part (a) we have $\mathcal{L}(y)=0$ is solved by $y(x) = \frac{c_1}{x} + c_2 x^2$, so we find c_1, c_2 in each IVP, noting $y'(x) = -\frac{c_1}{x^2} + 2c_2 x$

$$y(1) = c_1 + c_2 = 0$$
$$y'(1) = -c_1 + 2c_2 = -1$$

$$\Rightarrow \begin{cases} c_1 = \frac{1}{3} \\ c_2 = -\frac{1}{3} \end{cases}$$

$$y_L(x) = \frac{1/3}{x} - \frac{1}{3} x^2$$

$$\begin{cases} y(2) = \frac{c_1}{2} + 4c_2 = 2 \\ y'(2) = \frac{-c_1}{4} + 4c_2 = -1 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = 4 \\ c_2 = 0 \end{cases}$$

$$y_R(x) = \frac{4}{x}$$

④ ⓒ $G(x,s) = \begin{cases} A(s)\, y_L(x) & 1 \le x \le s \\ B(s)\, y_R(x) & s \le x \le 2 \end{cases}$

$\boxed{5}$

where $A(s)$ & $B(s)$ are defined as

$$A(s) = \frac{y_R(s)}{a_2(s) W(s)}, \quad B(s) = \frac{y_L(s)}{a_2(s) W(s)}.$$

Here, $a_2(s) = s^2$ and

$+1$ $$W(s) = y_L(s) y_R'(s) - y_R(s) y_L'(s)$$

$$= \left(\frac{1}{3x} - \frac{1}{3}x^2\right)\left(-\frac{4}{x^2}\right) - \frac{4}{x}\left(\frac{-1}{3x^2} - \frac{2}{3}x\right)$$

$$= \cancel{\frac{-4}{3x^3}} + \frac{4}{3} + \cancel{\frac{4}{3x^3}} + \frac{8}{3}$$

$$= 4$$

thus

$+1$ $$A(s) = \frac{4/s}{s^2 \cdot 4} = \frac{1}{s^3}$$

$+1$ $$B(s) = \frac{1/3s - \frac{1}{3}s^2}{s^2 \cdot 4} = \frac{1}{12s^3} - \frac{1}{12}$$

$+2$ $$\Rightarrow G(x,s) = \begin{cases} \frac{1}{s^3}\left(\frac{1}{3x} - \frac{x^2}{3}\right) & 1 \le x \le s \\ \frac{1}{3}\left(\frac{1}{s^3} - 1\right)\frac{1}{x} & s \le x \le 2. \end{cases}$$

④ ⓓ $y(x) = \int_1^2 G(x,s)\, e^s \, ds.$

④ ⓓ $y(x) = \int_1^2 G(x,s)\, e^s\, ds$ $+1$

$\boxed{5}$

$$= \int_1^x \frac{1}{3}\left(\frac{1}{x^3} - x\right)\frac{1}{s}\, e^s\, ds \quad +2$$

$$+ \int_x^2 \frac{1}{x^3}\left(\frac{1}{3s} - \frac{s^2}{3}\right) e^s\, ds \quad +2$$

Q4 a, b, c, d Bookwork / Modified from Coursework

(5) (a) $\begin{cases} \dot{x} = 4y \\ \dot{y} = -x \end{cases}$ has fixed points where

part (a)

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{cases} 4y = 0 \\ -x = 0 \end{cases}$ so there is one fixed point $(x(t), y(t)) = (0,0)$.

(5) (b) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ +1

$\boxed{5}$

$A = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}$ has eigenvalues λ given

by $\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} -\lambda & 4 \\ -1 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i \quad +2$$

Associated Eigenvectors will be complex conjugates so we find one of them

$\lambda = 2i$ has eigenvector satisfying

$$Av = \lambda v = 2iv \iff \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{cases} 4v_2 = 2i v_1 \\ -v_1 = 2i v_2 \end{cases}$$

choose $v_2 = 1 \implies v_1 = -2i$

$$\implies v_{\lambda_\pm} = \begin{pmatrix} \mp 2i \\ 1 \end{pmatrix} \quad +2$$

5 ⓑ The general solution to the system is

$\boxed{5}$ given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} v_{\lambda_1} + c_2 e^{\lambda_2 t} v_{\lambda_2}$$

+2

$$= c_1 e^{2it} \begin{pmatrix} -2i \\ 1 \end{pmatrix} + c_2 e^{-2it} \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

Imposing the initial conditions

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1 \begin{pmatrix} -2i \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

+1 $$\Rightarrow \begin{cases} a = 2i(c_2 - c_1) \\ b = c_1 + c_2 \end{cases}$$

+2 $$\Rightarrow \begin{cases} c_1 = \frac{1}{2}\left(b - \frac{a}{2i}\right) \\ c_2 = \frac{1}{2}\left(b + \frac{a}{2i}\right) \end{cases}$$

5 ⓒ To sketch the phase portait, we need

$\boxed{5}$ the real solution:

$$x(t) = 2i\left(-c_1 e^{2it} + c_2 e^{-2it}\right)$$

$$= 2i\left(-c_1 [\cos 2t + i \sin 2t] + c_2 [\cos 2t - i \sin 2t]\right)$$

+1

$$= 2i\left([c_2 - c_1]\cos 2t - i[c_1 + c_2]\sin 2t\right)$$

$$= 2i\left(\frac{a}{2i}\cos 2t - i b \sin 2t\right)$$

$$= a\cos 2t + 2b\sin 2t$$

+2 for any attempt in this direction

5 c (Continued)

$$y(t) = c_1 e^{2it} + c_2 e^{-2it}$$

$$= c_1[\cos 2t + i \sin 2t] + c_2[\cos 2t - i \sin 2t]$$

+1 $$= [c_1 + c_2] \cos 2t + i[c_1 - c_2] \sin 2t$$

$$= b \cos 2t + i\left(-\frac{a}{2i}\right) \sin 2t$$

$$= b \cos 2t - \frac{a}{2} \sin 2t$$

Notice that

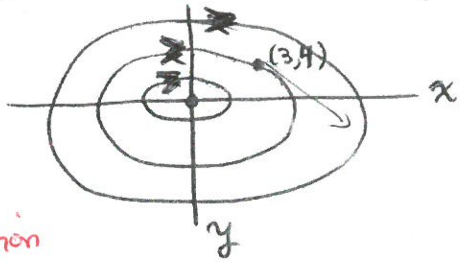
$$x^2 = a^2 \cos^2 2t + 4ab \cos 2t \sin 2t + 4b^2 \sin^2 2t$$

$$y^2 = b^2 \cos^2 2t - ab \cos 2t \sin 2t + \frac{a^2}{4} \sin^2 2t$$

+1 $$\Rightarrow x^2 + 4y^2 = a^2 + 4b^2$$

which are ellipses

+1



+1 The IC (3,4) has a unique ellipse passing through that point which satisfies the dynamical system.

no credit for wrong diagram with no explanation

5 d The zero solution is a centre and it is a stable equilibrium with

$$\lim_{t \to \infty} x(t) = 0.$$

+2 WHY? either from correct phase diagram or explained from eigenvalues

+1

$\boxed{5}$

+2 One fixed point: at $(x(t), y(t)) = (0,0)$ $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ when $\begin{cases} 4y = 0 \\ -x = 0 \end{cases}$

5 a, b, c, d Coursework / Modified Coursework

+1 if incorrect but justified via eigenvalues