Queen Mary
University of London
Final Exam Solutions
2020

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Problem 1

Similar as lecture examples

Let r be the per capita growth rate of a population in the time interval dt and N be the population density, which is the total number of individuals in this population. Note r is a constant number not a variable here.

a) *Solve the general solution to the first-order ordinary differential equation (ODE), $\frac{dN}{dt} = rN$. Supposing $r = 1$, find the specific solution when the initial population density at $t = 0$ is $N(0) = 100$.* (5 marks)

Solution: The ODE for the population density is

$$\frac{dN}{dt} = rN.$$

The general solution is $N = Ce^{rt}$, where D can be any arbitrary positive value (3 marks). When $t = 0, N(0) = C = 100$. Thus, the solution for the initial value problem is $N = 100e^{t}$(2 mark).

b) *Now suppose the per capita growth rate will decrease linearly with the population density. When the population density approaches its maximum size K, the per capita growth rate decrease to be 0. This yields the logistic equation,*

$$\frac{dN}{dt} = N(1 - \frac{N}{K}).$$

Find the general solution of this ODE by the method of separation variables. Note K is a constant number not a variable here. According to this general solution, describe how will the population size change when $t \to \infty$. (10 marks)

Solution: We can rewrite the equation as

$$\frac{dN}{N(1 - \frac{N}{K})} = 1dt$$

(2 marks)

$$\int \frac{K}{N(K - N)} dN = \int 1dt$$

(1 mark)

$$\int (\frac{1}{N} + \frac{1}{K-N})dN = t + C$$

(2 marks)

$$\ln N - \ln(K-N) = t + C$$
$$\frac{N}{K-N} = De^t$$
$$N = \frac{DKe^t}{De^t + 1},$$

where D can be any arbitrary positive value (2 marks). We can rewrite our general solution as $\frac{K}{1+\frac{1}{D}e^{-t}}$, thus when time goes infinity, the population size approaches K (3 marks).

Problem 2

similar form as tutorial problems with adaptations

a) *Find the general of the following ordinary differential equation*

$$(x-1)y' = 2y.$$

. (5 marks)

Solution: First we solve $y' = \frac{2y}{x-1}$ by separating variables. $H(y) = \ln|y|$, hence $H^{-1}(u) = \pm e^u$, and on the right-hand side $\int \frac{2}{x-1}\,dx = 2\ln|x-1| + C$. This gives

$$y = \pm e^{2\ln|x-1|+C} = D(x-1)^2$$

where we denote $D = \pm e^C$ (5 marks).

b) *Use the Picard-Lindelöf Theorem to show that* $0 < A < 1$ *is required for the ODE in (a) with the initial condition* $y(0) = 1$ *to have a unique solution in a rectangular domain* $|x| \leq A, |y-1| \leq B$. *Describe the position of this domain in the xy plane. Find out the other conditions between A and B to guarantee the uniqueness of the solution in this domain.* (8 marks)

Solution:
In our case of initial condition means $x = 0$ and $y(0) = 1$, which refers to a point $(0,1)$ in the xy plane. Hence, the rectangular domain $\mathcal{D} = (|x| \leq A,\ |y-1| \leq B)$ is a domain around point $(0,1)$ with the width as $2A$ and the height as $2B$ (1 mark). The right-hand side $f(x,y) = \frac{2y}{x-1}$ is continuous everywhere in $\mathcal{D}$, if $\mathcal{D}$ does not include $x = 1$. As $-A \leq x \leq A$, we require $0 < A < 1$ for the uniqueness of the solution inside $\mathcal{D}$ (2 mark).
In addition, we have the maximal value of $|f(x,y)|$ in domain $\mathcal{D}$ is $M = \max_{\mathcal{D}} |\frac{2y}{x-1}| =$

$\frac{2(1+B)}{|1-A|}$, when $x = A$ and $y = 1 + B$. The existence and uniqueness of the solution to the ODE requires $A \leq B/M$. Thus, we have

$$A \leq B/M = \frac{B|1-A|}{2(1+B)}.$$

(3 mark).

The partial derivative of $f(x, y)$ is bounded if domain $\mathcal{D}$ does not include $x = 1$, because

$$\left|\frac{\partial f}{\partial y}\right| = \frac{2}{x-1},$$

(2 marks), which is the same condition as $f(x, y)$ is continuous in this domain. Thus, to guarantee the uniqueness of solution in Domain $\mathcal{D}$ is $0 < A < 1$ and $A \leq \frac{B|1-A|}{2(1+B)}$.

c) *Use the Picard-Lindelöf Theorem to show whether the ODE in (a) has a unique solution around a different initial condition $y(1) = 0$. If not, based on the general solution obtained in (a) and this initial condition $y(1) = 0$, sketch and describe all possible solutions to this initial value problem in the xy plane. (7 marks)*

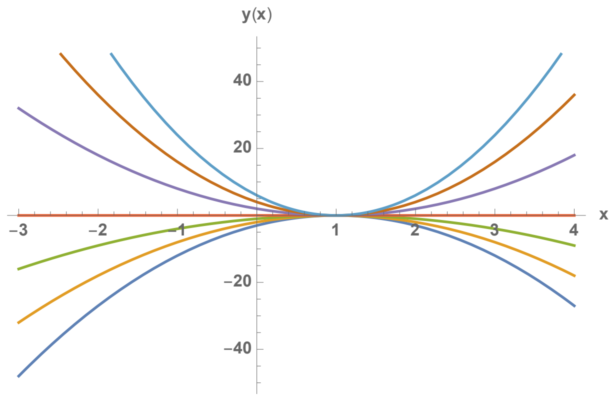
Solution:

The initial condition corresponds to the point $(1, 0)$ in the xy plane (1 mark). As the function $f(x, y) = \frac{2y}{x-1}$ is discontinuous at $x = 1$, according to the Picard-Lindelöf Theorem there is no rectangular domain around this initial point which can guarantee the uniqueness of the solution to the I.V.P. (3 marks)

The general solution of the ODE is $y = D(x-1)^2$. As $y(1) = D * 0 = 0$. Thus, D can be any arbitrary number. All solutions belongs to the general solution and their combinations can be a solution to this IVP, which can be written as

$$y(x) = \begin{cases} D_1(x-1)^2, & \text{if} \quad x \geq 1 \\ D_2(x-1)^2, & \text{if} \quad x < 1 \end{cases},$$

Where D_1 and D_2 can be any arbitrary real number which can equal or not. We can sketch the solutions as



. (4 marks)

Problem 3

similar form as tutorial problems with adaptations

a) *Find the general solution to the homogeneous second-order linear ODE*

$$2y'' + y' - 15y = 0.$$

(5 marks)

Solution: The ODE is second-order linear ODE with constant coefficients, thus we first write down the characteristic equation is $2\lambda^2 + \lambda - 15 = 0$ (2 marks). This equations has two real roots $\lambda_1 = 5/2, \lambda_2 = -3$. Hence, the general solution is $y_h = C_1 e^{\frac{5}{2}x} + C_2 e^{-3x}$ (3 marks).

b) *Use the solution in (a) and educated guess method to find the general solution to the inhomogeneous second-order linear ODE*

$$2y'' + y' - 15y = 6e^{-2x}.$$

(10 marks)

Solution:

First, we check whether -2 is a root of the characteristic equation of the corresponding homogeneous ODE. Since the function e^{-2x} is not a solution to the homogeneous equation, we may use the educated guess (2 marks).

Second, we look for the particular solution of the inhomogeneous equation in the form $y_p(x) = d_0 e^{-2x}$ (2 marks).

Thus we have $y_p'(x) = -2d_0 e^{-2x}$ (1 mark) and $y_p''(x) = 4d_0 e^{-2x}$ (1 mark). Substituting these back into the inhomogeneous equation gives on the left-hand side $(8-2-15)d_0 e^{-2x} = 6e^{-2x}$ so that to match to the right-hand side we should choose $d_0 = -2/3$, hence $y_p(x) = -\frac{2}{3}e^{-2x}$ (2 marks). (*If the students directly write the solution for d_0, they also get the full*

4 marks in this step.)
Finally, the general solution to the inhomogeneous equation is given by

$$y_g(x) = y_h(x) + y_p(x) = C_1 e^{\frac{5}{2}x} + C_2 e^{-3x} - \frac{2}{3} e^{-2x}$$

(2 marks).

c) *Use the variation of parameter method to find the general solution of the inhomogeneous equation*

$$y'' - 5y' + 6y = e^{3x} \cos x.$$

(10 marks)

Solution: According to the variation of parameter method, the general solution of the inhomogeneous equation will be $y_g(x) = y_h(x) + y_p(x)$, where $y_p(x)$ is a particular solution and $y_h(x)$ is the general solution of the corresponding homogeneous equation. (2 marks) The corresponding characteristic equation $\lambda^2 - 5\lambda + 6 = 0$ has two roots: $\lambda_1 = 2$ and $\lambda_2 = 3$ and the general solution is given by:

$$y_h(x) = C_1 e^{2x} + C_2 e^{3x}.$$

(3 marks)
As $\lambda_1 - \lambda_2 = 2 - 3 = -1$ we have $y_p(x) = -1 \cdot (e^{2x} \int e^{-2x} e^{3x} \cos x\, dx - e^{3x} \int e^{-3x} e^{3x} \cos x\, dx)$ $= -e^{2x} \int e^x \cos x\, dx + e^{3x} \int \cos x\, dx$, where $\int \cos x\, dx = \sin x$.
Using the formula provided in the appendix,

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x).$$

Giving

$$y_p(x) = -\frac{1}{2} e^{3x} (\sin x + \cos x) + e^{3x} \sin x = \frac{1}{2} e^{3x} (\sin x - \cos x)\ .(3\ marks)$$

Finally, the general the solution of the inhomogeneous differential equation is given by

$$y_g(x) = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{3x} (\sin x - \cos x)\ .$$

(2 marks)

Problem 4

similar form as tutorial problems with adaptations

a) *Write down the general solution to the following Euler-type second order differential equation*

$$x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = 0.$$

(9 marks)

Solution: According to the general method of solving the Euler-type equation we introduce the new variable by $x = e^t$ and the new function $z(t)$ so that

$$z(t) = y(e^t), \quad \Rightarrow \quad \frac{dz}{dt} = e^t y', \qquad \frac{d^2 z}{dt^2} = e^t y' + e^{2t} y''$$

From the above we find correspondingly that $y' = e^{-t}\dot{z}$, $y'' = e^{-2t}(\ddot{z} - \dot{z})$ (4 marks). Substituting to the Euler-type equation reduces the latter to a homogeneous equation with constant coefficients:

$$e^{2t} \cdot e^{-2t}(\ddot{z} - \dot{z}) - 4e^t \cdot e^{-t}\dot{z} + 2z = \ddot{z} - 5\dot{z} + 6z = 0,$$

(2 marks)

The corresponding characteristic equation $\lambda^2 - 5\lambda + 6 = 0$ has two roots: $\lambda_1 = 2$ and $\lambda_2 = 3$ and the general solution is given by:

$$z(t) = C_1 e^{2t} + C_2 e^{3t},$$

for arbitrary constants C_1 and C_2 (2 marks). Finally, substituting $t = \ln x$ gives

$$y(x) = C_1 x^2 + C_2 x^3,$$

(1 mark)

b) *Find the solution to the following Boundary Value Problem,*

$$y'' + 9y = 0, \; y'(0) = 5, \; y(\frac{\pi}{3}) = -\frac{5}{3}.$$

(6 marks)

Solution: The corresponding characteristic equation $\lambda^2 + 9\lambda = 0$ has two roots: $\lambda_1 = 3i$ and $\lambda_2 = -3i$ and the general solution is given by:

$$y(x) = C_1 \cos 3x + C_2 \sin 3x.$$

(3 marks)

Using the boundary conditions, we have

$$y(\frac{\pi}{3}) = C_1 \cos\frac{3\pi}{3} + C_2 \sin\frac{3\pi}{3} = -C_1 = -5/3$$

and

$$y'(0) = -3C_1 \sin 0 + 3C_2 \cos 0 = 3C_2 = 5.$$

Thus the solution of the BVP is $y(x) = \frac{5}{3}(\cos 3x + \sin 3x)$. (3 mark)

Problem 5

Similar as past resit exam paper with adaptations

Consider a system of two nonlinear first-order ODEs,

$$\dot{x} = xy - 4, \quad \dot{y} = (x-4)(y-x).$$

a) *Compute all equilibria of this ODE system.* (6 marks)

Solution.

There are in total 3 equilibria of this ODE system. The right-hand side $\dot{y} = (x - 4)(y - x) = 0$ for either $x = 4$ or $y = x$. (3 marks) For $x = 4$ the right-hand side $\dot{x} = xy - 4 = 0$ vanishes for $y = 1$, hence we have an equilibrium at the point $(4, 1)$ in the (x, y) plane (1 mark). For $y = x$ the right-hand side $\dot{x} = xy - 4 = 0$ for $x = y = \pm 2$, giving two more equilibria at the points $(2, 2)$ and $(-2, -2)$ (2 marks).

b) *Linearise the above equations around the equilibrium at $y = 2$ and write down its matrix form. Find the corresponding eigenvalues and eigenvectors to this linearised systems and write down its general solution. (10 marks)*

Solution. The equilibrium with $y = 2$ is at $(2, 2)$. To linearize around this point, we need to evaluate $\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}$ at the point of equilibrium, where $f_1 = xy - 4$ and $f_2 = (x - 4)(y - x)$. We obtain

$$\frac{\partial f_1}{\partial x} = y|_{x=2,y=2} = 2, \quad \frac{\partial f_1}{\partial y} = x|_{x=2,y=2} = 2,$$

$$\frac{\partial f_2}{\partial x} = (y + 4 - 2x)|_{x=2,y=2} = 2, \quad \frac{\partial f_2}{\partial y} = (x - 4)|_{x=2,y=2} = -2.(3\ marks)$$

Thus, we have the linearized system as

$$\dot{x} = 2x + 2y, \quad \dot{y} = 2x - 2y, \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(1 mark).
The characteristic equation is given by $(2 - \lambda)(-2 - \lambda) - 4 = \lambda^2 - 8 = 0$ with the two real roots of the opposite sign $\lambda_{1,2} = \pm 2\sqrt{2}$ (1 mark).
The eigenvector corresponding to $\lambda_1 = 2\sqrt{2}$ can be found from

$$\begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 2\sqrt{2} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \quad \Rightarrow p_1 = \frac{\sqrt{2}}{2 - \sqrt{2}} q_1 = (\sqrt{2} + 1) q_1$$

(2 marks).

so that the eigenvector can be chosen as $\mathbf{u}_1 = \begin{pmatrix} \sqrt{2} + 1 \\ 1 \end{pmatrix}$ by setting $q_1 = 1$(1 mark). Similarly, we have the eigenvector for $\lambda_2 = -2\sqrt{2}$, $\mathbf{u}_2 = \begin{pmatrix} -\frac{\sqrt{2}}{2+\sqrt{2}} \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$ (1 mark). We can see that these two eigenvectors are orthogonal as $\mathbf{u}_1\mathbf{u}_2 = (-\sqrt{2}+1)(\sqrt{2}+1)+1 = 0$.

(Note we can also write $\mathbf{u}_1 = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1 \\ -\sqrt{2}-1 \end{pmatrix}$, which will lead to the same result of general solution.)

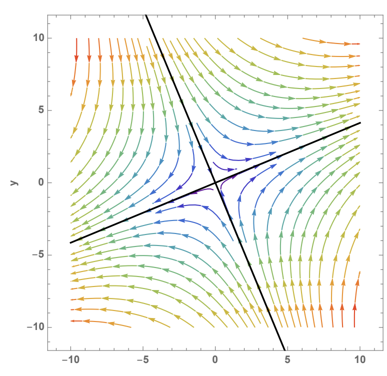
Thus, the general solution of this system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{(2\sqrt{2})t} \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} + C_2 e^{(-2\sqrt{2})t} \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$$

(2 marks).

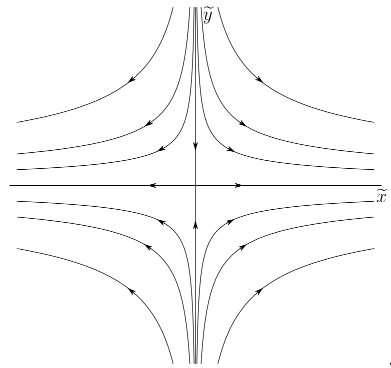
c) *Determine the type of equilibrium of the original ODE system at $y = 2$ (center, stable node sink, stable spiral, saddle, unstable node source, unstable spiral) and sketch its phase portrait for the linearised system in* (b). (4 marks)

Solution. As the real part of the eigenvalues are one positive and one negative, this critical point $(2, 2)$ is a saddle (2 marks). The phase portrait of this system around this saddle point in the orginal coordinates is



. (2 marks)

We can also introduce the vector $\tilde{\mathbf{x}} \equiv \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$ of new coordinates $\tilde{x}, \tilde{y}$ related to the vector of old coordinates $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ via $\tilde{\mathbf{x}} = U^{-1}\mathbf{x}$, where the columns of the 2×2 matrix U are chosen to be the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. or in the new coordinates $\tilde{x}, \tilde{y}$ is



d) *Using the result in (b), find the solution of this linearised system corresponding to the initial conditions $x(0) = 3 + \sqrt{2}$, $y(0) = 3$. Determine the tangent vector to the trajectory of the solution at $t = 0$ and the values of $x(t \to \infty)$ to this specified initial condition.* (5 marks)

Solution. From the general solution we have

$$y(t) = C_1 e^{(2\sqrt{2})t} + C_2 e^{(-2\sqrt{2})t}, \quad \Rightarrow \quad y(0) = C_1 + C_2 = 3$$

$$x(t) = (1+\sqrt{2})C_1 e^{(2\sqrt{2})t} + (1-\sqrt{2})C_2 e^{(-2\sqrt{2})t}, \quad \Rightarrow \quad x(0) = (C_1+C_2)+\sqrt{2}(C_1-C_2)$$
$$= 3 + \sqrt{2}(C_1 - C_2) = 3 + \sqrt{2}$$

which gives $C_1 = 2, C_2 = 1$. (2 marks).
(Note if using $\mathbf{u}_1 = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1 \\ -\sqrt{2}-1 \end{pmatrix}$, then $C_1 = 2+2\sqrt{2}, C_2 = 1-\sqrt{2}$.)
Hence the trajectory of the solution is given by

$$x(t) = 2(1+\sqrt{2})e^{(2\sqrt{2})t} + (1-\sqrt{2})e^{(-2\sqrt{2})t},$$

$$y(t) = 2e^{(2\sqrt{2})t} + e^{(-2\sqrt{2})t},$$

where both x and y increases over time and become positive infinity for $t \to \infty$ (1 mark). The components of the initial tangent vector determining the speed of increasing are given by $\dot{x}(0) = 2(\sqrt{2}+6)$, $\dot{y}(0) = 2\sqrt{2}$ (2 mark).