MTH5123 Differential Equations

Solution to the Exam Problems 2015

All problems are either already seen in example classes, or slight modifications of those.

1. **a)** Find the general solution of the homogeneous ODE $4y'' + 4y' + y = 0$ (6 points)

Solution: The characteristic equation is $4\lambda^2 + 4\lambda + 1 = (2\lambda + 1)^2 = 0$ [**1p**] which has a real root: $\lambda_1 = -1/2$ of multiplicity two [**2p**]. The general solution to the homogeneous equation is given by $y_h(x) = (c_1 + c_2 x)e^{-x/2}$ with arbitrary constants c_1 and c_2. [**3p**].

b) Find the general solution of the non-homogeneous ODE

$$4y'' + 4y' + y = \cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)$$

(12 points)

Solution: Since the functions $\cos\left(\frac{x}{2}\right)$ and $\sin\left(\frac{x}{2}\right)$ are not solutions to the homogeneous equation[**2p**], we may use the "educated guess" method and look for the particular solution of the non-homogeneous equation in the form $y_p(x) = A\cos\left(\frac{x}{2}\right) + B\sin\left(\frac{x}{2}\right)$[**2p**] so that:

$$y_p' = -\frac{A}{2}\sin\left(\frac{x}{2}\right) + \frac{B}{2}\cos\left(\frac{x}{2}\right), \quad y_p''(x) = -\frac{A}{4}\cos\left(\frac{x}{2}\right) - \frac{B}{4}\sin\left(\frac{x}{2}\right)$$

so that $y_p'' = -\frac{1}{4}y_p$, [**1p**]. Substituting this back to the nonhomogeneous equation gives in the left-hand side:

$$4y_p'' + 4y_p' + y_p = -y_p + 4y_p' + y_p = -2A\sin\left(\frac{x}{2}\right) + 2B\cos\left(\frac{x}{2}\right) \quad [\mathbf{2p}]$$

so that to match to the right-hand side we should choose $A = -1/2, B = 1/2$ so that $y_p(x) = -\frac{1}{2}\cos\left(\frac{x}{2}\right) + \frac{1}{2}\sin\left(\frac{x}{2}\right)$[**3p**]. Finally, the general solution to the non-homogeneous equation is given by the sum:

$$y_g(x) = (c_1 + c_2 x)e^{-x/2} - \frac{1}{2}\cos\left(\frac{x}{2}\right) + \frac{1}{2}\sin\left(\frac{x}{2}\right), \qquad [\mathbf{2p}]$$

c) Solve the following initial value problem

$$y' = \frac{y}{x} + x\,, \quad y(1) = 2$$

(7 points)

Solution. The ODE is linear non-homogenious of the first order. The corresponding homogeneous ODE $y' = \frac{y}{x}$ is separable and following the standard procedure we introduce in the left-hand side $H(y) = \int \frac{dy}{y} = \ln|y|$, hence solving $H(y) = u$ we find $y = \pm e^u = H^{-1}(u)$. In the right-hand side we have

$$\int \frac{dx}{x} = \ln|x| + C,$$

so that the general solution to the homogeneous equation is given by

$$y_h = H^{-1}(\ln|x| + C) = \pm e^C |x| = Dx, \quad [\mathbf{3p}]$$

where we denoted $D = \pm e^C$ the constant of arbitrary sign. According to the variation of parameters method we look for a solution of the non-homogeneous ODE in the form:

$$y = D(x)\,x, \quad \Rightarrow y' = D'\,x + D,$$

Substituting this back to the equation $y' = \frac{y}{x} + x$ we have

$$y' = D'\,x + D = \frac{y}{x} + x \equiv D + x$$

which implies

$$D'(x) = 1, \quad D(x) = x + C$$

which gives for the general solution of the non-homogeneous ODE

$$y_g(x) = x(x + C), \quad [\mathbf{3p}]$$

Finally, the given initial value $y(1) = 2 = 1 + C$ requires $C = 1$ so that the solution to the initial value problem is given by $y = x(x+1)$, [**1p**].

Alternative way:

Substitution $y = xz(x)$, so that $y' = z + xz'$ and we have

$$z + xz' = z + x, \quad \Rightarrow \quad z' = 1 \quad \Rightarrow \quad z = x + C \quad \Rightarrow \quad y = x(x + C)$$

and from the initial condition $C = 1$. FULL MARKS.

2. **(a)** **(i)** Find all functions $f(y)$ for which the following differential equation becomes exact:

$$\frac{dy}{dx} = -\frac{x^3 + f(y)}{6xy^2 + 5y^4}, \tag{1}$$

(5 points)

Solution: Denoting $P(x,y) = x^3 + f(y), \quad Q(x,y) = 6xy^2 + 5y^4$ [**2p**] we can rewrite the above equation in the standard form $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$. We then have $\frac{\partial P}{\partial y} = \frac{d}{dy}(f(y))$ whereas $\frac{\partial Q}{\partial x} = 6y^2$ [**1p**], hence the equation is exact only if $\frac{d}{dy}(f(y)) = 6y^2$ or equivalently $f(y) = 2y^3 + C$ [**2p**], with any constant C.

(ii) Suppose, $f(y)$ is chosen so that the equation (1) is exact and $f(1) = 0$. Solve (1) in implicit form. (10 points)

Solution: The condition $f(1) = 2 + C = 0$ makes us to choose $C = -2$, so that $f(y) = 2(y^3 - 1)$ [**2p**]. Then the general solution should be looked for in implicit form as $F(x,y) = Const$ where

$$F = \int P(x,y)\,dx = \int \left(x^3 + 2(y^3 - 1)\right)\,dx = \frac{x^4}{4} + 2x(y^3 - 1) + g(y), \quad [\mathbf{3p}]$$

where $g(y)$ is to be determined from the condition $Q = \frac{\partial F}{\partial y} = 6xy^2 + g'(y)$ [**1p**]. We therefore conclude that $g'(y) = 5y^4$ [**1p**] so that $g(y) = y^5$ [**2p**]. Thus the solution in implicit form is $\frac{x^4}{4} + 2x(y^3 - 1) + y^5 = Const$ [**1p**].

b) Consider the initial value problem (IVP)

$$\frac{dy}{dx} = 2x\sqrt{|y-1|} \equiv \begin{cases} 2x\sqrt{y-1}, & y \geq 1 \\ 2x\sqrt{1-y}, & y < 1 \end{cases}, \quad y(0) = b\,.$$

where b is a real parameter.
Find the value of the parameter b such that the corresponding IVP may have more than one solution and explain your choice. Confirm your choice by giving at least two different solutions of the IVP in the domain $y \geq 1$ for such a value of the parameter. (6 points)

Solution: The solution to IVP is unique if the function $f(x,y) = 2x\sqrt{|y-1|}$ is continuous in some domain in the xy plane centered at the point with coordinates $x = 0, y = b$ [**1p**] and the modulus of its partial derivative $|\frac{\partial f}{\partial y}| = \frac{|x|}{\sqrt{|y-1|}}$

is bounded in the same domain [**1p**]. The second condition is certainly violated for the choice $b = 1$ so for that choice we may expect non-uniqueness. [**1p**]. Solving the differential equation by the separation of variables method we get for $y \geq 1$ the general solution:

$$\int \frac{dy}{2\sqrt{y-1}} = \int x\,dx, \quad \Rightarrow \sqrt{y-1} = \frac{x^2}{2} + C\,[\mathbf{1p}]$$

The initial condition $y(0) = 1$ fixes $C = 0$, so that a solution to IVP is $y = 1 + \frac{x^4}{4}$ [**1p**]. On the other hand the constant solution $y(x) = 1$ solves the same IVP [**1p**].

c) Consider the inhomogeneous boundary value problem (BVP)

$$y'' + b^2 y = 5 \quad , \quad y(0) = 1,\, y'\left(\frac{\pi}{2}\right) = 1\,.$$

where $b > 0$ is a real parameter. Find all positive values of the parameter b such that the corresponding BVP may have either no solution or infinitely many solutions. (4 points)

Solution: From the Theorem of the Alternative the inhomogeneous boundary value problem may have no solutions or infinitely many solutions if and only if the corresponding homogeneous boundary value problem

$$y'' + b^2 y = 0 \quad , \quad y(0) = 0,\, y'\left(\frac{\pi}{2}\right) = 0\,.$$

allows a non-zero solution[**1p**]. The characteristic equation for the associated ODE is $\lambda^2 + b^2 = 0$ with two complex conjugate roots $\lambda = \pm ib$ for $b > 0$. The general solution is then given by

$$y(x) = A\cos(bx) + B\sin(bx)$$

The left homogeneous BC $y(0) = 0$ implies that $A = 0$ so that $y(x) = B\sin(bx)$[**1p**], then $y'(x) = Bb\cos(bx)$ so that the right-end BC gives $y'\left(\frac{\pi}{2}\right) = Bb\cos\frac{\pi b}{2} = 0$[**1p**] which is possible for $\frac{\pi b}{2} = \frac{\pi}{2}(2n+1)$ for $n = 0, 1, 2, ...$, hence $b = 1, 3, 5...$ - any positive odd integer[**1p**].

3. Write down the solution to the following Boundary Value Problem (BVP) for the second order non-homogeneous differential equation

$$\frac{1}{(x+1)}\frac{d^2y}{dx^2} - \frac{1}{(x+1)^2}\frac{dy}{dx} = f(x), \quad y(0) = 0,\ y'(1) = 0$$

by using the Green's function method along the following lines:

a) Show that the left-hand side of the ODE can be written down in the form $\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right)$ for some function $r(x)$ and use this fact to determine the general solution of the associated homogeneous ODE. (4 points)

Solution: We have

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) = r(x)\frac{d^2y}{dx^2} + r'(x)\frac{dy}{dx}$$

which indeed coincides with the original ODE for $r(x) = \frac{1}{x+1}$. The homogeneous ODE therefore has the form:

$$\frac{d}{dx}\left(\frac{1}{x+1}\frac{dy}{dx}\right) = 0 \quad [\mathbf{1p}]$$

and can be easily integrated to find the general solution [**3p**]:

$$\frac{1}{x+1}\frac{dy}{dx} = C_1, \quad \Rightarrow \quad \frac{dy}{dx} = C_1(x+1), \quad \Rightarrow \quad y(x) = C_1\left(\frac{x^2}{2} + x\right) + C_2$$

for arbitrary constants C_1 and C_2.

b) Formulate the corresponding left-end and right-end initial value problems and use their solutions to construct the Green's function $G(x,s)$. (14 points)

Solution: The left-end boundary condition $y(0) = 0$ is imposed at $x_1 = 0$. By comparing it to the standard form $\alpha y'(x_1) + \beta y(x_1) = 0$ we conclude that $\alpha = 0, \beta = 1$ [**1p**]. Then the left-end initial value problem for the function $y_L(x)$ is formulated as

$$y_L(x_1) = \alpha, y_L'(x_1) = -\beta, \quad \Rightarrow \quad y_L(0) = 0, y_L'(0) = -1 \quad [\mathbf{2p}]$$

Substituting here the general solution of the homogeneous equation yields $C_2 = 0$, $C_1 = -1$ [**1p**] so that

$$y_L(x) = -\left(\frac{x^2}{2} + x\right), \qquad [\mathbf{1p}]$$

Obviously, $x_2 = 1$ and by comparing the right-end boundary condition $y'(1) = 0$ to the standard form $\gamma y'(x_2) + \delta y(x_2) = 0$ we conclude that $\gamma = 1, \delta = 0$[**1p**]. Then the right-end initial value problem for the function $y_R(x)$ is formulated as

$$y_R(x_2) = \gamma, y_R'(x_2) = -\delta, \quad \Rightarrow \quad y_R(1) = 1, y_R'(1) = 0, \quad [\mathbf{1p}]$$

which now gives $C_1 = 0$, $C_2 = 1$ and finally

$$y_R(x) = 1, \qquad [\mathbf{2p}]$$

Now we can use $y_L(x), y_R(x)$ for constructing the Green's function $G(x,s)$. First we calculate the Wronskian

$$W(s) = y_L(s)y_R'(s) - y_R(s)y_L'(s) = (s+1), \quad [\mathbf{1p}]$$

We also should take into account that from the original ODE $a_2(s) = \frac{1}{s+1}$ so that $a_2(s)W(s) = 1$ [**1p**] and we have

$$A(s) = y_R(s)/\left(a_2(s)W(s)\right) = 1, \quad B(s) = y_L(s)/\left(a_2(s)W(s)\right) = -\left(\frac{s^2}{2} + s\right), \quad [\mathbf{1p}]$$

Finally the Green's function is constructed as

$$G(x,s) = \begin{cases} A(s)y_l(x), & 0 \le x \le s \\ B(s)y_R(x), & s \le x \le 1 \end{cases}$$

$$= \begin{cases} -\left(\frac{x^2}{2} + x\right), & 0 \le x \le s \\ -\left(\frac{s^2}{2} + s\right), & s \le x \le 1 \end{cases}, \quad [\mathbf{2p}]$$

c) Write down the solution to the BVP in terms of $G(x,s)$ and $f(x)$ and use it to find the explicit form of the solution for $f(x) = 2x$.

(7 points)

Solution: The solution to the boundary value problem is given by

$$y(x) = \int_0^1 G(x,s)\, f(s)\, ds = \int_0^x G(x,s)\, f(s)\, ds + \int_x^1 G(x,s)\, f(s)\, ds$$

$$= -\int_0^x \left(\frac{s^2}{2} + s\right) f(s)\, ds - \left(\frac{x^2}{2} + x\right)\int_x^1 f(s)\, ds\,, \quad [\mathbf{2p}]$$

substituting here $f(x) = 2x$ and using

$$\int_0^x \left(\frac{s^2}{2} + s\right) 2s\, ds = \left[\frac{2}{3}s^3 + \frac{1}{4}s^4\right]_0^x = \frac{2}{3}x^3 + \frac{1}{4}x^4, \quad [\mathbf{1p}]$$

and

$$\int_x^1 2s\, ds = s^2|_x^1 = 1 - x^2, \quad [\mathbf{1p}]$$

we finally can write the solution in the form

$$y(x) = -\left[\frac{1}{4}x^4 + \frac{2}{3}x^3 + (1 - x^2)\left(\frac{x^2}{2} + x\right)\right], \quad [\mathbf{1p}]$$

which after simplifying yields

$$y(x) = \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} - x, \quad [\mathbf{2p}]$$

4. Consider

$$\dot{x} = 2x - 4y, \quad \dot{y} = ax - 6y\,. \tag{2}$$

where $-\infty < a < \infty$ is a real parameter.

a) For the particular value $a = -5$ determine eigenvalues and eigenvectors associated with the system, find equations for stable and unstable invariant manifolds and sketch the phase portrait.

(11 points)

Solution. First for $a = -5$ we rewrite the system in the matrix form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ where the matrix A associated with the system is given by $A = \begin{pmatrix} 2 & -4 \\ -5 & -6 \end{pmatrix}$ [**1p**] The characteristic equation is $(2-\lambda)(-6-\lambda) - 20 = \lambda^2 + 4\lambda - 32 = 0$ and has two real roots $\lambda_1 = 4$ and $\lambda_2 = -8$, [**1p**]. Eigenvector corresponding to $\lambda_1 = 4$ is found as

$$\begin{pmatrix} 2 & -4 \\ -5 & -6 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 4 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \quad [\mathbf{2p}]$$

which implies $4q_1 = -2p_1$, [**1p**] hence we can choose for example $p_1 = 2$ and $q_1 = -1$. For second eigenvalue $\lambda_2 = -8$ we similarly find

$$\begin{pmatrix} 2 & -4 \\ -5 & -6 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = -8 \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}, \quad [\mathbf{2p}]$$

which implies $4q_2 = 10p_2$, [**1p**] so that we can choose, for example $p_2 = 2, q_2 = 5$.

As $\lambda_1 > 0$ the trajectories will be for $t \to +\infty$ parallel to the straight line (the "unstable manifold") given by $y = \frac{q_1}{p_1}x = -\frac{1}{2}x$ [**1p**], whereas for $t \to -\infty$ they will be parallel to the stable manifold $y = \frac{q_2}{p_2}x = \frac{5}{2}x$ [**1p**]. The corresponding phase portrait can be sketched as [**1p**]:

Sketch to be placed here

Description: a diagram of two intersecting invariant manifolds: $y = -\frac{1}{2}x$ (with the arrow showing motion along it away from the origin) and $y = \frac{5}{2}x$ (with the arrow showing motion along it towards the origin)and separating the plane in 4 quadrants. The rest is a bunch of trajectories which are hyperbolas starting tangent to $y = \frac{5}{2}x$ in all quadrants and flowing finally tangent to $y = -\frac{1}{2}x$.

b) Classify for which values of the parameter a the equilibrium point $x = y = 0$ of that system represents (ii) focus (ii) node and (iii) saddle. For which values of the parameter a the equilibrium is not hyperbolic?

(9 points)

Solution. For a general a we rewrite the system in the matrix form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ where $A = \begin{pmatrix} 2 & -4 \\ a & -6 \end{pmatrix}$. The characteristic equation is now $(2 - \lambda)(-6-\lambda) + 4a = \lambda^2 + 4\lambda + 4a - 12 = 0$ with the roots

$$\lambda_{1,2} = \frac{1}{2}\left(-4 \pm \sqrt{64 - 16a}\right) = -2 \pm 2\sqrt{4-a}\,[\mathbf{1p}]\,.$$

We see that for $a > 4$ the eigenvalues are complex conjugate with negative real part:

$$\lambda_{1,2} = -2 \pm 2i\sqrt{a-4}\,[\mathbf{1p}]\,.$$

hence for $a > 4$ the equilibrium is a stable focus [**1p**].

For $a < 4$ both eigenvalues are real [**1p**]. Whereas $\lambda_2 = -2 - 2\sqrt{4-a} < 0$ for all $a < 4$, the eigenvalue $\lambda_1 = -2 + 2\sqrt{4-a}$ may change sign. Solving $\lambda_1 = 0$ gives $a = 3$ (thus for such a the equilibrium is not hyperbolic [**1p**]) and we further see that $\lambda_1 > 0$ for $a < 3$ and $\lambda_1 < 0$ for $3 < a < 4$ [**2p**]. We therefore conclude that for $3 < a < 4$ we have both $\lambda_1 < 0$ and $\lambda_2 < 0$, so the equilibrium is a stable node [**1p**]. For $a < 3$ the eigenvalues have different signs and the equilibrium is a saddle[**1p**].

c) Consider a system of two nonlinear first-order ODE:

$$\dot{x} = -x^3 + 2y^3, \quad \dot{y} = -2xy^2\,. \tag{3}$$

Demonstrate how to use the function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ to investigate the stability of the above system. (5 points)

Solution. The function $V(x, y) = \frac{1}{2}(x^2 + y^2) > 0$ for $(x, y) \neq (0, 0)$ [**1p**] and its orbital derivative is given by

$$\mathcal{D}_f V = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= x(-x^3+2y^3)+y\left(-2xy^2\right) = -x^4+2xy^3-2xy^3 = -x^4 \leq 0, \forall\,(x, y) \neq (0, 0) \quad [\mathbf{3p}]$$

Therefore $V(x, y)$ is a valid Lyapunov function ensuring the stability (but not asymptotic stability) of the solution of nonlinear equation in the whole (x, y) plane [**1p**].