MTH5123 Differential Equations

Most of the problems are unseen but are similar to/variations of those considered in weekly tutorials/lectures/mock exam paper.
The students are supposed to make an active use of the provided **Formula Sheet**.

Solution to the Exam Problems 2014

1. **a)** Find a function $f(u)$ such that the differential equation

$$f(x+y) + \ln x + \left(e^{x+y} + y^2\right)\frac{dy}{dx} = 0$$

is exact (5 points)

Solution: Denoting

$$P(x,y) = f(x+y) + \ln x, \quad Q(x,y) = e^{x+y} + y^2$$

we have $\frac{\partial P}{\partial y} = f'(x+y)$ whereas $\frac{\partial Q}{\partial x} = e^{x+y}$. The equation is exact only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ which gives $f'(x+y) = e^{x+y}$. In turn this implies $f(u) = e^u + const$, and we can take a particular case $const = 0$.

b) For the chosen $f(u)$ write down the corresponding solution in implicit form. (11 points)

Solution: For such a choice of $f(x+y) = e^{x+y}$ the general solution can be looked for in implicit form $F(x,y) = C$ where

$$F(x,y) = \int P(x,y)\,dx = \int \left(e^{x+y} + \ln x\right)\,dx = e^{x+y} + x\ln|x| - x + g(y)$$

where $g(y)$ is to be determined from the condition $Q = \frac{\partial F}{\partial y} = e^{x+y} + g'(y)$. We conclude that $g'(y) = y^2$, hence $g(y) = \frac{y^3}{3} + const$, and the general solution is given in implicit form as

$$e^{x+y} + x\ln|x| - x + \frac{y^3}{3} = C$$

c) *almost identical to the problem considered in a weekly tutorial*
Consider the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad f(x,y) = \sqrt{y^2+9}, \quad y(1) = 0$$

Show that the Picard-Lindelöf Theorem guarantees the uniqueness and existence of the solution to the above problem in a rectangular domain $\mathcal{D} = (|x-a| \le A,\ |y-b| \le B)$ in the xy plane, and specify the parameters a and b. Find the possible range of values of the height B of the domain $\mathcal{D}$ given that the width A of the domain satisfies $A < 1/2$. (9 points)

Solution: the right-hand side $f(x,y) = \sqrt{9+y^2}$ is continuous everywhere, and its derivative $\frac{\partial f}{\partial y}$ satisfies $\left|\frac{\partial f}{\partial y}\right| = |y|/\sqrt{9+y^2} < 1$, so is bounded. The initial conditions imply $a = 1$ and $b = y(1) = 0$, hence in the rectangular domain $\mathcal{D} = (|x-1| < A,\ |y| < B)$ the solution to ODE exists and is unique, provided $A < B/M$, with $M = \max_{\mathcal{D}} \sqrt{y^2+9}$. The function $f(x,y) = \sqrt{9+y^2}$ in the right-hand side of the ODE obviously grows with $|y|$ so for a given B its maximum M is achieved for $|y| = B$. We then have $M = \sqrt{9+B^2}$ which implies that the width A should satisfy $A < B/M = B/\sqrt{9+B^2}$. The maximal value of the width is $A = B/\sqrt{9+B^2}$ and requiring $A < 1/2$ we have

$$B/\sqrt{9+B^2} < 1/2, \quad \Rightarrow (2B)^2 < 9 + B^2, \quad \Rightarrow B^2 < 3$$

Thus, for $B < \sqrt{3}$ the width of the region where the uniqueness and existence is guaranteed satisfies $A < 1/2$.

2. **a)** Find the general solution of the homogeneous ODE $y'' + 9y = 0$ (5 points)

Solution: The characteristic equation is $\lambda^2 + 9 = 0$ which has two complex conjugate roots: $\lambda_1 = 3i$ and $\lambda_2 = -3i$. The general solution to the homogeneous equation can be written in the form $y_h(x) = c_1 \cos 3x + c_2 \sin 3x$.

b) Find the general solution of the non-homogeneous ODE

$$y'' + 9y = \sin(2x)$$

. (11 points)

Solution: Since the function $\sin 2x$ is not a solution to the homogeneous equation, we may use the "educated guess" method and look for the particular solution of the nonhomogeneous equation in the form $y_p(x) = A \sin 2x + B \cos 2x$ so that $y_p' = 2A \cos 2x - 2B \sin 2x$ and $y_p'' = -4y_p$. Substituting this back to the non-homogeneous equation gives in the left-hand side: $(A \sin 2x + B \cos 2x)(-4+9) = 5A \sin 2x + 5B \cos 2x$. To match to the right-hand side we should choose $A = 1/5$, $B = 0$ so that $y_p(x) = \frac{1}{5} \sin 2x$. Finally, the general solution to the nonhomogeneous equation is given by the sum:

$$y(x) = c_1 \sin 3x + c_2 \cos 3x + \frac{1}{5} \sin 2x$$

c) *c)+d) are identical to a problem considered in one of weekly tutorials*

Write down the general solution to the first order homogeneous linear ODE

$$y' = \tan(x)\, y$$

(5 points)

Solution. The homogeneous ODE $y' = \tan(x)y$ is separable and following the standard procedure we introduce in the left-hand side $H(y) = \int \frac{dy}{y} = \ln|y|$, hence solving $H(y) = u$ we find $y = \pm e^u = H^{-1}(u)$. In the right-hand side we have

$$\int \tan x \, dx = -\ln|\cos x| + C$$

so that the general solution to the homogeneous equation is given by

$$y_h = H^{-1}\left(-\ln|\cos x| + C\right) = \pm e^C \frac{1}{|\cos x|} = D\frac{1}{\cos x}$$

where we denoted $D = \pm e^C$ the constant of arbitrary sign.

d) Solve the initial value problem for the first order linear non-homogeneous ODE

$$y' = \tan(x)\, y + 1, \quad y(0) = 2\,.$$

(4 points)

Solution. *The standard methods used in the lectures was the variation of parameters, but some students prefer the integrating factor method. They will be given full marks if arrive to the correct solution.*

According to the variation of parameters method we look for a solution of the non-homogeneous ODE in the form:

$$y = \frac{D(x)}{\cos x}, \quad \Rightarrow y' = \frac{D'(x)}{\cos x} + D(x)\frac{\sin x}{\cos^2 x}$$

Substituting this back to the equation $y' = \frac{\sin(x)}{\cos(x)}\, y + 1$ we have

$$\frac{D'(x)}{\cos x} + D(x)\frac{\sin x}{\cos^2 x} = \frac{D(x)}{\cos x}\tan x + 1$$

which implies

$$D'(x) = \cos x, \qquad \Rightarrow D(x) = \sin x + C$$

which gives for the general solution of the non-homogeneous ODE

$$y_g(x) = \frac{1}{\cos x}\left(\sin x + C\right)$$

As $y(0) = C = 2$, we finally find the solution to the initial value problem

$$y_g(x) = \frac{1}{\cos x}\left(\sin x + 2\right) = \tan x + \frac{2}{\cos x}$$

The problem is a variation of one considered in weekly tutorials

3. Write down the solution to the following Boundary Value Problem (BVP) for the second order nonhomogeneous differential equation

$$\frac{d^2y}{dx^2} = f(x), \quad y(0) = 0,\ y'(1) = 0$$

by using the Green's function method along the following lines:

a) Formulate the corresponding left-end initial value problem and find its solution $y_L(x)$. (8 points)

Solution. The general solution $y_g(x)$ to the linear homogeneous equation $y'' = 0$ is easily found to be given by

$$y_g(x) = c_1 x + c_2$$

The left-end boundary condition $y(0) = 0$ is imposed at $x_1 = 0$. By comparing it to the standard form $\alpha y'(x_1) + \beta y(x_1) = 0$ we conclude that $\alpha = 0, \beta = 1$. Then the left-end initial value problem for the function $y_L(x)$ is formulated as

$$y_L(x_1) = \alpha, y_L'(x_1) = -\beta, \quad \Rightarrow \quad y_L(0) = 0, y_L'(0) = -1$$

so that $c_2 = 0,\ c_1 = -1$ which immediately gives

$$y_L(x) = -x$$

b) Formulate the corresponding right-end initial value problem and find its solution $y_R(x)$. (7 points)

Solution. Obviously, $x_2 = 1$ and by comparing the right-end boundary condition $y'(1) = 0$ to the standard form $\gamma y'(x_2) + \delta y(x_2) = 0$ we conclude that $\gamma = 1, \delta = 0$. Then the right-end initial value problem for the function $y_R(x)$ is formulated as

$$y_R(x_2) = \gamma, y_R'(x_1) = -\delta, \quad \Rightarrow \quad y_R(1) = 1, y_R'(1) = 0$$

which now gives $c_1 + c_2 = 1,\ c_1 = 0$ so that

$$y_R(x) = 1$$

c)) Use $y_L(x), y_R(x)$ for constructing the Green's function $G(x,s)$. (6 points)

Solution. Using $y_L(x)$ and $y_R(x)$ we calculate the Wronskian

$$W(s) = y_L(s)y_R'(s) - y_R(s)y_L'(s) = 0 + 1 = 1$$

so that

$$A(s) = 1, \quad B(s) = -s$$

Finally the Green's function is constructed as

$$G(x,s) = \begin{cases} -A(s)x, & 0 \le x \le s \\ B(s), & s \le x \le 1 \end{cases}$$

$$= \begin{cases} -x, & 0 \le x \le s \\ -s, & s \le x \le 1 \end{cases}$$

d) Write down the solution to the BVP. in terms of $G(x,s)$ and $f(x)$ and use it to find the explicit form of the solution for $f(x) = x^2$. (4 points) **Solution.** The solution to the boundary value problem is given by

$$y(x) = \int_0^1 G(x,s)\, f(s)\, ds = \int_0^x G(x,s)\, f(s)\, ds + \int_x^1 G(x,s)\, f(s)\, ds$$

$$= -\int_0^x s\, f(s)\, ds - x\int_x^1 f(s)\, ds$$

substituting here $f(x) = x^2$ gives

$$y(x) = -\int_0^x s^3\, ds - x\int_x^1 s^2\, ds = -\frac{x^4}{4} - x\frac{1}{3}(1 - x^3) = -\frac{x}{3} + \frac{x^4}{12}$$

4. Consider a system of two nonlinear first-order ODE:

$$\dot{x} = -x - 3y - 3x^3, \quad \dot{y} = \frac{4}{3}x - y - \frac{1}{3}x^3 \tag{1}$$

a) Write down in the matrix form the system obtained by linearization of the above equations around the point $x = y = 0$ and find the corresponding eigenvalues and eigenvectors. (8 points)

Solution. Discarding nonlinear terms we arrive at

$$\dot{x} = -x - 3y, \quad \dot{y} = \frac{4}{3}x - y, \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ \frac{4}{3} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic equation is given by $(-1-\lambda)^2 + 4 = 0$ with two complex-conjugate roots $\lambda_{1,2} = -1 \pm 2i$. The eigenvector corresponding to $\lambda_1 = -1 + 2i$ can be found from

$$\begin{pmatrix} -1 & -3 \\ \frac{4}{3} & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = (-1+2i) \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \quad \Rightarrow q_1 = -\frac{2}{3} i p_1$$

so that the eigenvector can be chosen as $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -\frac{2}{3}i \end{pmatrix}$. As second eigenvector $\mathbf{u}_2$ must be the complex conjugate of $\mathbf{u}_1$ we can immediately write down $\mathbf{u}_2 = \begin{pmatrix} 1 \\ \frac{2}{3}i \end{pmatrix}$

b) Write down general solution of the linear system. Discuss the stability of zero solution of such a linear system and determine the value $x(t \to \infty)$. (4 points)

Solution. As the real part of the eigenvalues is negative the system is asymptoticallly stable which implies $x(t) \to 0$ as $t \to \infty$. This can be also inferred directly from the general solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{(-1+2i)t} \begin{pmatrix} 1 \\ -\frac{2}{3}i \end{pmatrix} + C_2 e^{(-1-2i)t} \begin{pmatrix} 1 \\ \frac{2}{3}i \end{pmatrix}$$

c) Find the solution of the linear system corresponding to the initial conditions $x(0) = 2$, $y(0) = 0$. Determine the type of equilibrium for the system and describe in words the shape of trajectory in the phase plane corresponding to the specified initial conditions. Determine the tangent vector to the trajectory at $t = 0$. (8 points)

Solution. From the general solution we have

$$x(t) = C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t}, \quad \Rightarrow \quad x(0) = C_1 + C_2 = 2$$

$$y(t) = -\frac{2}{3} i \left(C_1 e^{(-1+2i)t} - C_2 e^{(-1-2i)t} \right), \quad \Rightarrow \quad y(0) = -\frac{2}{3} i (C_1 - C_2) = 0$$

which gives $C_1 = C_2 = 1$. Hence the trajectory is given by coordinates

$$x(t) = e^{(-1+2i)t} + e^{(-1-2i)t} = 2e^{-t} \cos 2t$$

$$y(t) = -\frac{2}{3} i \left(e^{(-1+2i)t} - e^{(-1-2i)t} \right) = \frac{4}{3} e^{-t} \sin 2t$$

which has the shape of a spiral rotating around the origin and approaching it asymptotically for $t \to \infty$. The type of equilibrium is a stable focus. The components of the initial tangent vector determining the direction of rotation are given by $\dot{x}(0) = -2$, $\dot{y}(0) = 8/3$.

d) Demonstrate how to use the function $V(x, y) = \frac{4}{3} x^2 + 3y^2$ to investigate the stability of the full non-linear system (**??**). (5 points)

Solution. $V(x, y) > 0$ for $x \neq 0, y \neq 0$ and $V(0, 0) = 0$. We also have the orbital derivative:

$$\mathcal{D}_f V = \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial y} \dot{y}$$

$$= \frac{8}{3} x(-x - 3y - 3x^3) + 6y \left(\frac{4}{3} x - y - \frac{1}{3} y^3 \right) = -\frac{8}{3} x^2 - 6y^2 - 8x^4 - 2y^4 < 0$$

for any $(x, y) \neq (0, 0)$. Therefore $V(x, y)$ is a valid Lyapunov function ensuring the stability of the solution of nonlinear equation.