MTH5123 Differential Equations

Lecture Notes

Week 2

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2. Consider equations of the type

$$y' = F\left(\frac{y}{x}\right) \tag{1.8}$$

Such ODEs do not change if we rescale $x \to kx$ and $y \to ky$ for any real constant factor $k \neq 0$, hence they are known under the name **scale-invariant** first order ODEs. To reduce them to separable equations one introduces a new function $z(x) = y(x)/x$ which implies $y(x) = xz(x)$. Differentiating this equation gives $y' = z(x) + xz'(x)$, and (1.8) can be rewritten in the form $z + xz' = F(z)$ or equivalently

$$z' = \frac{1}{x}\left[F(z) - z\right] \tag{1.9}$$

which is indeed separable.

Example:

Solve the equation

$$xy' = y - xe^{y/x}.$$

Solution:

After dividing both sides by x we see that the equation is of the form (1.8) with the right-hand side $F(z) = z - e^z$. Therefore it is equivalent to the separable equation

$$z' = -\frac{1}{x}e^z\,.$$

Solving it by standard means leads to $e^{-z} = \ln|x| + C$ or

$$z = -\ln\left(\ln|x| + C\right)\,.$$

Finally the general solution to the original ODE is

$$y(x) = -x\ln\left(\ln|x| + C\right)\,.$$

1.3 First order linear ODEs

This class of equations is given by

$$y' = A(x)\,y + B(x)\,, \tag{1.10}$$

where the two functions $A(x) \neq 0$ and $B(x)$ are known. These equations are called **linear** (in y), because y and its derivative y' occur only to the first power, they are not multiplied together, nor do they appear as the argument of a function (such as $\sin y, \exp(y)$, etc.). If $B(x) = 0$, the equation is called **homogeneous**, if $B(x) \neq 0$ it is called **inhomogeneous**.

Example:

$y' = \sin(x)y$ homogeneous
$y' = e^x y + x$ inhomogeneous
$y' = 1 - y^2 + x$ nonlinear

The method of solution of such equations proceeds in two steps:

Step 1: Solve the *homogeneous* equation $y' = A(x)\,y$, which is separable. The general solution is found to be

$$\int \frac{dy}{y} = \int A(x)\,dx + C \Rightarrow \ln|y| = \int A(x)\,dx + C \tag{1.11}$$

and finally

$$y = De^{\int A(x)\,dx}, \tag{1.12}$$

where $D \in \mathbb{R}$ is an arbitrary real constant (also called a free parameter).

Step 2 is known as the **variation of parameter** method. It amounts to looking for the solution of (1.10) in the form

$$y = D(x)\,e^{\int A(x)\,dx}, \tag{1.13}$$

where $D(x)$ is now an unknown function to be determined by substituting (1.13) to (1.10). This gives

$$y' = D'(x)\,e^{\int A(x)\,dx} + A(x)D(x)\,e^{\int A(x)\,dx} = A(x)D(x)\,e^{\int A(x)\,dx} + B(x)\,,$$

which after cancelling equal terms on both sides is equivalent to

$$D'(x)\,e^{\int A(x)\,dx} = B(x). \tag{1.14}$$

This allows us to write $D'(x) = e^{-\int A(x)\,dx}\,B(x)$ and to recover $D(x)$ by simple integration

$$D(x) = \int e^{-\int A(x)\,dx}\,B(x)\,dx + C \tag{1.15}$$

finally yielding the general solution of (1.10) in the form

$$y(x) = e^{\int A(x)\,dx}\left(\int e^{-\int A(x)\,dx}\,B(x)\,dx + C\right) \quad \forall C \in \mathbb{R} \tag{1.16}$$

Note: An alternative method to derive the same result is the *integrating factor method*, as you have seen in Calculus 2.

Example:
Solve the equation

$$y' + 2xy = x.$$

Solution:
First we solve $y' + 2xy = 0$ by separation of variables obtaining $y = De^{-x^2}$, where D is an arbitrary constant. Now we assume $D = D(x)$ and substitute $y = D(x)e^{-x^2}$ to the full non-homogeneous equation:

$$y' = D'(x)e^{-x^2} + D(x)(-2x)e^{-x^2}.$$

Thus, we have

$$D'(x)e^{-x^2} + D(x)(-2x)e^{-x^2} + 2xD(x)e^{-x^2} = x$$

which implies $D'(x) = xe^{x^2}$, hence $D(x) = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$. Finally, the general solution to the original ODE is given by

$$y(x) = \left(\frac{1}{2}e^{x^2} + C\right)e^{-x^2} = \frac{1}{2} + Ce^{-x^2}.$$

1.4 Exact first order ODEs.

Exact ODEs are of the form

$$P(x,y) + Q(x,y)\,\frac{dy}{dx} = 0\,. \tag{1.17}$$

We would like to find solutions of this class of ODEs in *implicit form* $F(x,y) = C\,,\ y = y(x)$, for a constant C. Using the chain rule we observe that

$$\frac{dF(x,y(x))}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\,\frac{dy}{dx} = 0\,, \tag{1.18}$$

which coincides with (1.17) if we define

$$P(x,y) = \frac{\partial F}{\partial x}, \quad Q(x,y) = \frac{\partial F}{\partial y}\,. \tag{1.19}$$

Using these definitions we have

$$\frac{\partial}{\partial y}P(x,y) = \frac{\partial^2 F}{\partial y \partial x}, \quad \frac{\partial}{\partial x}Q(x,y) = \frac{\partial^2 F}{\partial x \partial y}. \tag{1.20}$$

If F is twice differentiable in both x and y with continuous second order partial derivatives, we have (according to the *mixed derivatives theorem* in Calculus 2)

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}\,,$$

and we conclude that the equation

$$\frac{\partial}{\partial y}P(x,y) = \frac{\partial}{\partial x}Q(x,y) \tag{1.21}$$

must hold. Equation (1.21) is the crucial condition for (1.17) to be **exact**. For any exact ODE the general solution can always be written in the implicit form $F(x,y) = C$.
To determine the form of the function $F(x,y)$, one may start with the first equation in (1.19) by integrating it over the variable x to

$$P(x,y) = \frac{\partial F}{\partial x} \quad \Rightarrow F(x,y) = \int P(x,y)dx + g(y)\,, \tag{1.22}$$

where the function $g(y)$ is an arbitrary function of the variable y, yet to be determined. To find $g(y)$ we use the second equation in (1.19)

$$Q(x,y) = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int P(x,y)dx + g'(y)\,, \tag{1.23}$$

which gives

$$g'(y) = Q(x,y) - \frac{\partial}{\partial y}\int P(x,y)dx \tag{1.24}$$

The missing function $g(y)$ can then be found by straightforward integration of this equation.

Example:
Show that the equation

$$3x^2 + y - (3y^2 - x)\frac{dy}{dx} = 0$$

is exact and find its general solution in implicit form.

Solution:
We identify $P(x,y) = 3x^2 + y$, hence $\frac{\partial P}{\partial y} = 1$. Similarly, $Q(x,y) = -(3y^2 - x)$, hence $\frac{\partial Q}{\partial x} = 1$. Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ the equation is exact.
We find its implicit solution in the form $F(x,y) = C$ by

$$F(x,y) = \int P(x,y)dx + g(y) = \int (3x^2 + y)\, dx + g(y) = x^3 + xy + g(y)\ ,$$

where $g(y)$ is yet undetermined. We further have

$$\frac{\partial F}{\partial y} = x + g'(y) = Q(x,y) = -(3y^2 - x), \quad \Rightarrow g'(y) = -3y^2\ .$$

This allows us to find

$$g(y) = \int (-3y^2)\, dy = -y^3 + C_1\ ,$$

where C_1 is an arbitrary constant. There is no need to keep C_1, as it can always be absorbed into the constant C. The general solution of the original equation in implicit form is obtained as

$$F(x,y) = x^3 + xy - y^3 = C.$$

Note:
The same ODE can be presented in a different form, for example:

$$\frac{dy}{dx} = \frac{3x^2 + y}{3y^2 - x}$$

One needs to recognize the equivalence of this equation to the form of an exact ODE by then applying the same procedure for a solution.