Queen Mary
University of London

MTH5123 Differential Equations

Lecture Notes

Week 10

School of Mathematical Sciences
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4.3.2 Phase portrait shape when $D = (\mathbf{Tr}A)^2 - 4\det A > 0 \,\&\, \det A < 0$: saddle-type (unstable)

But first we generalize our analysis: The last example fulfills $D = (\mathrm{Tr}A)^2 - 4\det A > 0$ for which the matrix A has two distinct real eigenvalues. For systems with $D > 0$ specific coordinates can always be defined as in this example: Having two real linearly independent eigenvectors $\mathbf{u}_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \neq \mathbf{0}$ and $\mathbf{u}_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} \neq \mathbf{0}$ of the matrix A allows us to build a non-singular matrix $U = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}$, $\det U \neq 0$, which can be inverted giving U^{-1}. We then change the coordinates (y_1, y_2) into new coordinates $(\tilde{y_1}, \tilde{y_2})$ via the transformation $\tilde{\mathbf{y}} = U^{-1}\mathbf{y}$ such that the system of two ODEs in the new coordinates takes the form $\frac{d}{dt}\tilde{\mathbf{y}} = \tilde{A}\tilde{\mathbf{y}}$ with $\tilde{A} = U^{-1}AU$ generalizing (4.16). As $\tilde{A}$ was shown to be diagonal, $\tilde{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, the two equations of the system defined by $\tilde{A}$ are now uncoupled and can be solved straightforwardly. Assuming the initial conditions $\tilde{y_1}(0) = \tilde{a}, \tilde{y_2}(0) = \tilde{b}$ we arrive at $\tilde{y_1}(t) = \tilde{a}e^{\lambda_1 t}$, $\tilde{y_2}(t) = \tilde{b}e^{\lambda_2 t}$. We see that the asymptotic behaviour and the phase portrait is determined by the *signs of the (real) eigenvalues* λ_1 and λ_2: If they are of different signs, $\lambda_2 < 0 < \lambda_1$, we have $\tilde{y_2} \to 0$ $(t \to \infty)$ whereas $\tilde{y_1} \to \infty$ for $\tilde{a} > 0$ and $\tilde{y_1} \to -\infty$ for $\tilde{a} < 0$ for $(t \to \infty)$. Nearby trajectories are given by *hyperbolic curves*

$$\tilde{y_2} = \tilde{b}\left(\frac{\tilde{y_1}}{\tilde{a}}\right)^{\lambda_2/\lambda_1} \tag{4.21}$$

if $\tilde{a} \neq 0$ and by the straight line $\tilde{y_1} = 0$ if $\tilde{a} = 0$. After this transformation the phase portrait qualitatively always looks like the one in Fig.4.1 (right), which is called a **saddle**. In the original coordinates the saddle will retain its topology, but generally it may be rotated and distorted; see Fig. 4.2 for two examples.

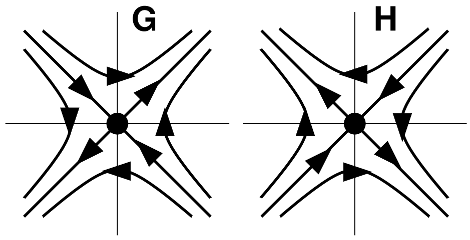

Figure 4.2: Two examples of phase portraits of saddle-type.

Example:
Find the general solution of the system of ODEs

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \;,\; A = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \tag{4.22}$$

and sketch the trajectories in phase space.

Solution:

We find (exercise for you) $\lambda_1 = 1$, $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = 2$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. According to (4.19) the solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

The initial conditions $y_1(0) = a, y_2(0) = b$ yield $a = c_1 + c_2, b = c_1 - c_2$. Solving for c_1, c_2 we get $c_1 = (a+b)/2, c_2 = (a-b)/2$. The explicit expressions for the time dependence of the coordinates in the (y_1, y_2) plane are thus given by

$$x = \frac{a+b}{2} e^t + \frac{a-b}{2} e^{2t} , \; y = \frac{a+b}{2} e^t - \frac{a-b}{2} e^{2t} .$$

4.3.3 Phase portrait shapes when $D = (\mathbf{Tr}A)^2 - 4\det A > 0 \& \det A > 0$: unstable node ($\mathbf{Tr}A > 0$) and stable ($\mathbf{Tr}A < 0$) node

How do the trajectories look like for different initial conditions? We start with $b = a$, which implies $y_1 = ae^t$, $y_2 = ae^t$. We conclude that $y_2 = y_1 \; \forall t$, i.e., this trajectory corresponds to motion along a straight line with slope one away from the origin, since both x and y tend to infinity when $t \to \infty$. Similarly, for $b = -a$ we have $y_2 = -y_1 = -a\, e^{2t} \; \forall t$, which describes motion along a straight line away from the origin. These two special straight lines, which intersect at the origin and partition the (y_1, y_2) plane into four sectors, define the invariant manifolds for the present system; see Fig. 4.3. Trajectories that start from initial conditions away from the invariant manifolds tend to be asymptotically parallel to one of the manifolds $y_2 = -y_1$ for $t \to \infty$, whereas close to the origin they are tangent to the other manifold $y_2 = y_1$. Such a phase portrait is topologically non-equivalent to the previous case.

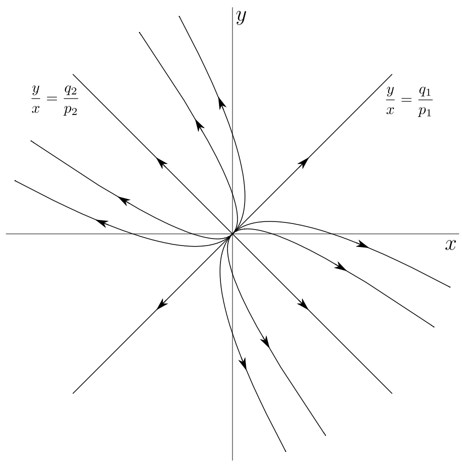

Figure 4.3: Phase portrait of the system (4.22) for which $q_1/p_1 = 1$, $q_2/p_2 = -1$.

As before, we now generalize our analysis: The last example still fulfills $D > 0$ like the previous one, but while in the first example the two eigenvalues had *different* signs, in the second one they have the *same* sign. In this case one can again perform a transformation into new coordinates $\tilde{y_1}, \tilde{y_2}$ exactly as before, and the trajectories are again given by (4.21) $\tilde{y_2} = \tilde{b}\left(\frac{\tilde{y_1}}{\tilde{a}}\right)^{\lambda_2/\lambda_1}$. However, due to both eigenvalues having the same sign in this case the curves are no longer hyperbolic. Instead, for $0 > \lambda_1 > \lambda_2$ they look like a set of parabolas tangent to the horizontal axis $\tilde{y_2} = 0$ at the origin of the $(\tilde{y_1}, \tilde{y_2})$ plane while for $\lambda_1 > \lambda_2 > 0$ they are tangent to the vertical axis $\tilde{y_1} = 0$; see Fig. 4.4 for examples.

The direction of motion along these curves is towards the origin if the eigenvalues are both negative, in which case the phase portrait is called a **stable node**. If the eigenvalues are both positive we have an **unstable node** with motion away from the origin; see again Fig. 4.4. Finally, if the initial conditions are chosen on the coordinate axis $\tilde{y_1} = 0$ or $\tilde{y_2} = 0$, the trajectory will coincide with the corresponding axis, which in turn defines an invariant manifold.

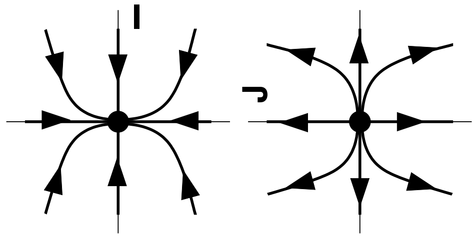

Figure 4.4: Phase portraits for a stable node with $0 > \lambda_1 > \lambda_2$ (left) and an unstable node with $\lambda_1 > \lambda_2 > 0$ (right).

In the original coordinates (y_1, y_2) the corresponding phase portraits retain these main features. Here the role of the invariant manifolds will be played by two straight lines intersecting at the origin defined by the corresponding eigenvectors, $y_2 = \frac{q_1}{p_1} y_1$ and $y_2 = \frac{q_2}{p_2} y_1$; see Fig. 4.3.

4.3.4 Phase portrait shape when $D = (\mathbf{Tr}A)^2 - 4\det A < 0$: Centre (stable, Tr$A = 0$), Spiral in (stable, Tr$A < 0$) and Spiral out (unstable, Tr$A > 0$)

Now we consider the second general case $D = (\mathrm{Tr}A)^2 - 4\det A < 0$ starting with another example.

Example:
Find the general solution of the system of ODEs

$$\dot{y_1} = y_2 \,, \; \dot{y_2} = -2y_1 + 2y_2$$

and visualize the trajectory which corresponds to the initial conditions $y_1(0) = 0\,, \, y_2(0) = 1$.

Solution:

Rewriting the system in matrix form we find $A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$. Its eigenvalues and eigenvectors were already given in example (ii) on p.5 of the lecture notes of week 9 as $\lambda_1 = 1+i$, $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ and $\lambda_2 = 1-i$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$. Since the eigenvectors are linearly independent, according to (4.19) the general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{(1+i)t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} + c_2 e^{(1-i)t} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \tag{4.23}$$

or equivalently

$$y_1 = e^t \left(c_1 e^{it} + c_2 e^{-it} \right) \,,\; y_2 = e^t \left(c_1 (1+i) e^{it} + c_2 (1-i) e^{-it} \right) \,.$$

The coefficients c_1, c_2 are determined by the above initial values,

$$0 = c_1 + c_2,\; 1 = c_1(1+i) + c_2(1-i) \;\Rightarrow\; c_1 = \frac{1}{2i},\; c_1 = -\frac{1}{2i} \,.$$

This leads to the trajectory

$$y_1 = e^t \sin t \,,\; y_2 = e^t (\sin t + \cos t) \,,$$

which describes a spiral in the form of a rotation in the (y_1, y_2) plane around the origin with period π, with the distance to the origin increasing exponentially in time. For example, the trajectory crosses the vertical axis $y_1 = 0$ periodically at times $t_n = 0, \pi, 2\pi, \ldots, \pi n, \ldots$, and the coordinates of the points of intersections are given by $(-1)^n e^{t_n}(0,1)$. Similarly, the trajectory intersects the diagonal $y_1 = y_2$ periodically at times $t_n^* = \frac{\pi}{2}, \frac{3}{2}\pi, \ldots, \left(n + \frac{1}{2}\right)\pi, \ldots$, and the coordinates of points of intersections are given by $(-1)^n e^{t_n^*}(1,1)$. We can find the direction of the tangent vector to the trajectory at $t = 0$ from the system of ODEs combined with the initial conditions yielding $\dot{y_1}(0) = y_2(0) = 1$, $\dot{y_2}(0) = -2y_1(0) + 2y_2(0) = 2$. Hence, the trajectory starts pointing towards the direction $(1,2)$. The resulting trajectory is sketched in Fig. 4.5.

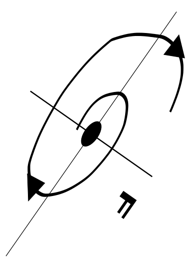

Figure 4.5: Sketch of the trajectory solving the initial value problem in the above example by spiraling away from the origin (please ignore the tilted axes).

Again we generalize our analysis for systems with $D < 0$ where the characteristic equation has two complex conjugate roots,

$$\lambda_{1,2} = r \pm i\omega \; , \; r = \frac{1}{2} Tr(A) \; , \; \omega = \sqrt{\det A - r^2} \, . \tag{4.24}$$

As we will see in a moment, the problem simplifies if we change variables $(x, y) \to (\tilde{y_1}, \tilde{y_2})$ by using the linear transformation

$$\tilde{\mathbf{y_1}} = W \mathbf{y_1} \; , \; W = \begin{pmatrix} a_{21} & r - a_{11} \\ 0 & \omega \end{pmatrix} \, . \tag{4.25}$$

In the new coordinates $(\tilde{y_1}, \tilde{y_2})$ the system of ODEs takes the form

$$\frac{d}{dt} \tilde{\mathbf{y_1}} = \tilde{A} \tilde{\mathbf{y_1}} \; , \; \tilde{A} = W A W^{-1} = \begin{pmatrix} r & -\omega \\ \omega & r \end{pmatrix} \, , \tag{4.26}$$

After this new transformation the matrix $\tilde{A}$ is not diagonal. Its eigenvalues and eigenvectors can be found to $\lambda_1 = r + i\omega$ with eigenvector $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\lambda_2 = r - i\omega$ with $\mathbf{u}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$. According to (4.19) we can write the general solution to (4.26) as

$$\begin{pmatrix} \tilde{y_1} \\ \tilde{y_2} \end{pmatrix} = c_1 e^{(r+i\omega)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(r-i\omega)t} \begin{pmatrix} 1 \\ i \end{pmatrix} \, . \tag{4.27}$$

The values of c_1 and c_2 can be determined by the two initial values $\tilde{y_1}(0) = \tilde{a}$ and $\tilde{y_2}(0) = \tilde{b}$, which gives $\tilde{a} = c_1 + c_2, \tilde{b} = -i(c_1 - c_2)$ leading to $c_1 = \frac{\tilde{a}+i\tilde{b}}{2}, c_2 = \frac{\tilde{a}-i\tilde{b}}{2}$. Using Euler's formula $e^{\pm i\omega t} = \cos \omega t \pm i \sin \omega t$ we obtain the solution for given initial conditions

$$\tilde{y_1} = e^{rt} \left(\tilde{a} \cos(\omega t) - \tilde{b} \sin(\omega t) \right) \; , \; \tilde{y_2} = e^{rt} \left(\tilde{b} \cos(\omega t) + \tilde{a} \sin(\omega t) \right) \, . \tag{4.28}$$

These two equations can be combined to yield the identity

$$\tilde{y_1}^2 + \tilde{y_2}^2 = e^{2rt} (\tilde{a}^2 + \tilde{b}^2) \, , \tag{4.29}$$

which gives us the key for the phase portrait of the system.

Centre (stable) phase portrait

First suppose that $r = 0$. In this case every trajectory starting at $t = 0$ from the point $(\tilde{a}, \tilde{b})$ in the plane is represented by a circle of radius $\sqrt{\tilde{a}^2 + \tilde{b}^2}$ centered at the origin. The fixed point in such a phase portrait is called a **centre**; see Fig. 4.6 (left). In original coordinates circular trajectories are generally deformed into ellipses in the plane, see Fig. 4.6 (right). The arrows show the direction of rotation along the ellipses with increasing time. They can be inferred from the initial tangent vector $\dot{\mathbf{y}} = (\dot{y_1}(0), \dot{y_2}(0))^T$.

Stable focus (spiral in and stable) phase portait

If $r < 0$ we see that asymptotically both $\tilde{y_1} \to 0$ $(t \to \infty)$ and $\tilde{y_2} \to 0$ $(t \to \infty)$. Hence, every trajectory approaches the origin along a spiral of shrinking radius.

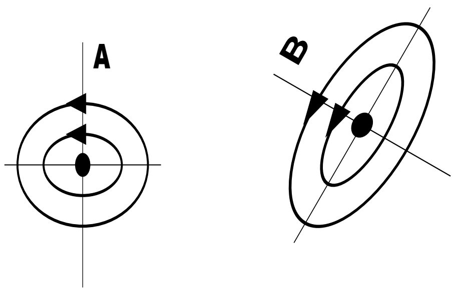

Figure 4.6: Phase portrait for a centre-type fixed point in transformed coordinates (left) and in original coordinates (right).

Unstable focus (spiral out and unstable) phase portrait
Similarly, if $r > 0$ the trajectory spirals away from the origin with ever increasing radius.

These two types of phase portraits are known as a **stable focus**, and respectively an **unstable focus**; see Fig. 4.7 for examples. Again, in original coordinates (y_1, y_2) the phase portraits retain their topological features, but the trajectories may be distorted, i.e., circular spirals may deform into an elliptic type, as we have already seen in the last example.

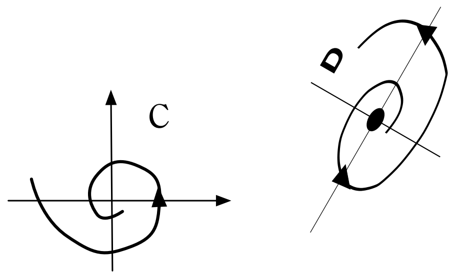

Figure 4.7: Phase portraits of fixed points of focus type: a stable one (left) and a distorted unstable one (right).

The only remaining case is $D = 0$, i.e., $(\mathrm{Tr}A)^2 = 4 \det A$ where both eigenvalues are equal and real, $\lambda_1 = \lambda_2 = \lambda = \frac{a_{11}+a_{22}}{2}$. This is a specific case yielding a special type of nodes, and in this module we will not dwell upon its analysis. This completes our classification of the different types of phase portraits in autonomous systems of two linear ODEs. With more advanced mathematical techniques it is possible to prove that the phase portrait of a nonlinear system (4.5) in the *vicinity* of an isolated fixed point is *topologically equivalent* to the phase portrait of the corresponding linear approximation **provided the real parts of all eigenvalues are nonzero**. Some information about this fact will be given in the next chapter.