Queen Mary
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MTH5123 Differential Equations

Formative Assessment Week 4 – Selected Solutions G. Bianconi

I. Practice Problems

A. Find the general solutions of the following linear homogeneous differential equations of second order:

1) $y'' + y' - 12y = 0$: The characteristic equation is $\lambda^2 + \lambda - 12 = 0$ which has two real roots $\lambda_1 = 3, \lambda_2 = -4$, hence the general solution is $y_h = C_1 e^{3x} + C_2 e^{-4x}$.

2) $6y'' + 5y' - 6y = 0$: The characteristic equation is $6\lambda^2 + 5\lambda - 6 = 0$ which has two real roots $\lambda_1 = 2/3, \lambda_2 = -3/2$, hence the general solution is $y_h = C_1 e^{\frac{2}{3}x} + C_2 e^{-\frac{3}{2}x}$.

3) $y'' + 2y' + 17y = 0$: The characteristic equation is $\lambda^2 + 2\lambda + 17 = 0$ which has two complex conjugate roots $\lambda_1 = -1 + 4i, \lambda_2 = -1 - 4i$, hence the general solution can be written as $y_h = e^{-x}\left(C_1 e^{4ix} + C_2 e^{-4ix}\right)$ where $C_{1,2}$ are two complex constants, or equivalently as $y_h = e^{-x}\left(A\cos(4x) + B\sin(4x)\right)$ where A, B are two real constants.

4) $y''+2y'+3y = 0$: The characteristic equation is $\lambda^2+2\lambda+3 = 0$ which has two complex conjugate roots $\lambda_1 = -1 + i\sqrt{2}, \lambda_2 = -1 - i\sqrt{2}$, hence the general solution can be written as $y_h = e^{-x}\left(C_1 e^{i\sqrt{2}x} + C_2 e^{-i\sqrt{2}x}\right)$ where $C_{1,2}$ are two complex constants or equivalently as $y_h = e^{-x}\left(A\cos(\sqrt{2}x) + B\sin(\sqrt{2}x)\right)$ where A, B are two real constants.

5) $16y'' + 8y' + y = 0$: The characteristic equation is $16\lambda^2 + 8\lambda + 1 = 0$ which has a real root $\lambda = -1/4$ of multiplicity two, hence the general solution can be written as $y_h = e^{-x/4}(C_1 + C_2 x)$.

B. Solve the following initial value problems:

1) $10y''-y'-3y = 0, \quad y(0) = 1, y'(0) = 0$: The characteristic equation is $10\lambda^2-\lambda-3 = 0$ which has two real roots $\lambda_1 = 3/5, \lambda_2 = -1/2$, hence the general solution is $y_h = C_1 e^{\frac{3}{5}x} + C_2 e^{-\frac{1}{2}x}$.

Taking the derivative: $y_h' = \frac{3}{5}C_1 e^{\frac{3}{5}x} - \frac{1}{2}C_2 e^{-\frac{1}{2}x}$. We then have $y(0) = C_1 + C_2 = 1$, $y'(0) = \frac{3}{5}C_1 - \frac{1}{2}C_2 = 0$. One can solve it by various methods. The most general method (although perhaps not the easiest) is to rewrite the system of linear equations in the matrix form as $A\mathbf{c} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 1 \\ \frac{3}{5} & -\frac{1}{2} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and solve it as

$$\mathbf{c} = A^{-1}\mathbf{b} = \frac{1}{-\frac{1}{2} - \frac{3}{5}} \begin{pmatrix} -\frac{1}{2} & -1 \\ -\frac{3}{5} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{11} \\ \frac{6}{11} \end{pmatrix}$$

so that $C_1 = \frac{5}{11}$, $C_2 = \frac{6}{11}$. Finally, the solution to the initial value problem is given by $y = \frac{5}{11}e^{\frac{3}{5}x} + \frac{6}{11}e^{-\frac{1}{2}x}$.

2) $y'' - 2y' - 3y = 0, \quad y(0) = 2,\ y'(0) = -3$: The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$ which has two real roots $\lambda_1 = 3, \lambda_2 = -1$, hence the general solution is $y_h = C_1e^{3x} + C_2e^{-x}$. Taking the derivative: $y_h' = 3C_1e^{3x} - C_2e^{-x}$. We then have $y(0) = C_1 + C_2 = 2$, $y'(0) = 3C_1 - C_2 = -3$. One can solve these equations by various methods to find $C_1 = -\frac{1}{4}$, $C_2 = \frac{9}{4}$. Finally, the solution to the initial value problem is given by $y = -\frac{1}{4}e^{3x} + \frac{9}{4}e^{-x}$.

3) $y'' - 4y' - 5y = 0, \quad y(0) = -1,\ y'(0) = -1$: The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$ which has two real roots $\lambda_1 = 5, \lambda_2 = -1$, hence the general solution is $y_h = C_1e^{5x} + C_2e^{-x}$. Taking the derivative: $y_h' = 5C_1e^{5x} - C_2e^{-x}$. We then have $y(0) = C_1 + C_2 = -1$, $y'(0) = 5C_1 - C_2 = -1$. One can solve these equations by various methods to find $C_1 = -\frac{1}{3}$, $C_2 = -\frac{2}{3}$. Finally, the solution to the initial value problem is given by $y = -\frac{1}{3}e^{5x} - \frac{2}{3}e^{-x}$.

4) $y'' - 4y' + 13y = 0, \quad y(0) = 4,\ y'(0) = 0$ The characteristic equation is $\lambda^2 - 4\lambda + 13 = 0$ which has two complex conjugate roots $\lambda_1 = 2 + 3i, \lambda_2 = 2 - 3i$, hence the general solution is $y_h = C_1e^{(2+3i)x} + C_2e^{(2-3i)x}$ or equivalently $y_h = e^{2x}(A\cos(3x) + B\sin(3x))$ where A, B are two real constants. Taking the derivative:

$$\begin{aligned} y_h' &= 2e^{2x}(A\cos(3x) + B\sin(3x)) + e^{2x}(-3A\sin(3x) + 3B\cos(3x)) \\ &= e^{2x}((2A + 3B)\cos(3x) + (2B - 3A)\sin(3x)) \end{aligned}$$

Hence $y(0) = A = 4$ and $y'(0) = 2A + 3B = 0$, so that $A = 4$, $B = -8/3$ and $y = e^{2x}(4\cos(3x) - \frac{8}{3}\sin(3x))$.

C. Assign to each of the following linear homogeneous differential equations

1) $2y'' - 8y' + 8y = 0$ **2)** $y'' + y' - 2y = 0$ **3)** $y'' + 2y' + 2y = 0$

a correct solution from the list:

i) $y = e^{-x}\left(2\cos x - \sqrt{2}\sin x\right)$ **ii)** $y = e^{x} + \frac{1}{7}e^{-2x}$ **iii)** $y = e^{2x}(x + 1)$.

Solution:

1) $2y'' - 8y' + 8y = 0$ corresponds to $y = e^{2x}(x+1)$ (characteristic equation is $2\lambda^2 - 8\lambda + 8 =$

$2(\lambda-2)^2=0$ with a real root of multiplicity two: $\lambda_1=\lambda_2=2$).

2) $y''+y'-2y=0$ corresponds to $y=e^x+\frac{1}{7}e^{-2x}$ (characteristic equation is $\lambda^2+\lambda-2=0$ with two real roots $\lambda_1=1, \lambda_2=-2$).

3) $y''+2y'+2y=0$ corresponds to $y=e^{-x}\left(2\cos x-\sqrt{2}\sin x\right)$ (characteristic equation is $\lambda^2+2\lambda+2=0$ with a pair of complex-conjugate roots $\lambda_1=-1+i, \lambda_2=-1-i$).

D. Determine the general solution for the homogeneous linear differential equation

$$y''-2y'+y=0\,.$$

Fix the constants of integration by the initial condition $y(2)=1,\ y'(2)=-2$ and write down the explicit form of the corresponding solution to the initial value problem.

Solution: The characteristic equation is $\lambda^2-2\lambda+1=0$. It has a single root $\lambda=1$ of multiplicity two. Hence the general solution is given by $y_h=(C_1x+C_2)e^x$, which gives after differentiation $y_h'=(C_1x+C_1+C_2)e^x$. The initial conditions give $y(2)=(2C_1+C_2)e^2=1,\ y'(2)=(3C_1+C_2)e^2=-2$ The easiest way to solve this system is to subtract the first equation from the second one, which gives $C_1e^2=-3$ so that $C_1=-3/e^2$ and from the first equation $C_2e^2=1-2C_1e^2=1+6=7$, hence $C_2=7/e^2$. The explicit solution to the initial value problem is $y=(-3x+7)e^{x-2}$.

III. More Practice with 2nd Order Linear ODEs

A. In each exercise below, solve the initial value problem and determine the value of α (if any) so that the solution approaches zero as $t\longrightarrow\infty$. Sketch/Graph the solution curve.

1) $\ddot{y}+5\dot{y}+6y=0$, $y(0)=\alpha$, $\dot{y}(0)=3$: Any α will work, since $\lambda=-2,-3$. One possible choice for α is $\alpha=2$, which gives $y(t)=9e^{-2t}-7e^{-3t}$.

3) $\ddot{y}+(2\alpha-1)\dot{y}+\alpha(\alpha-1)y=0$: $y(t)\longrightarrow 0$ as long as $\alpha>1$.

B. Study the equation $a\ddot{y}+b\dot{y}+cy=f$, where a,b,c *and* f are all constants. The equilibrium solution is $y_{eq}=f/c$ and the differential equation solved by $Y=y-y_{eq}$ (the deviation from equilibrium) is given by $a\ddot{Y}+b\dot{Y}+cY=0$.