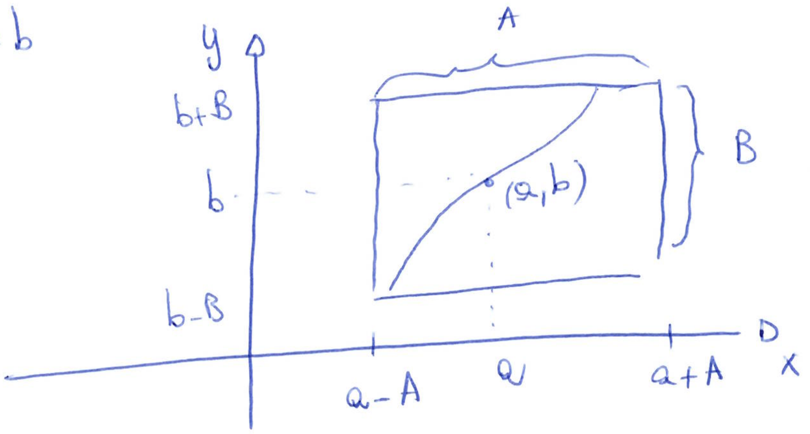
Initial Value Problem (IVP)

A given initial value problem (IVP) for 1^{st}-order ODE comprises

ODE: $y' = f(x, y)$

I.C. $y(a) = b$



The Picard-Lindelöf theorem provides the SUFFICIENT condition for the existence and uniqueness of the solution to the I.V.P. in the rectangular region D

$|x-a| \leq A$, $|y-b| \leq B$ for $A > 0$, $B > 0$.

Picard - Lindelöf theorem

Consider the IVP

$$y' = f(x,y) \qquad \& \qquad y(a) = b$$

This IVP has one and only one solution in a rectangular region D of the (x,y) plane defined by $|x-a| \le A$, $|y-b| \le B$ with $A > 0$, $B > 0$ provided the following conditions hold:

- The function $f(x,y)$ is continuous in D and therefore banded with

$$|f(x,y)| \le M \qquad \forall (x,y) \in D$$

for $M > 0$

- M is related to A and B as we must impose

$$\boxed{A \le \frac{B}{M}}$$

- Lipschitz condition

The partial derivative $\frac{\partial}{\partial y} f(x,y)$ is bounded in D

that is $$K = \max_{(x,y)\in D} \left| \frac{\partial f(x,y)}{\partial y} \right|$$

with K finite $\quad 0 < K < \infty$

K is called the Lipschitz constant

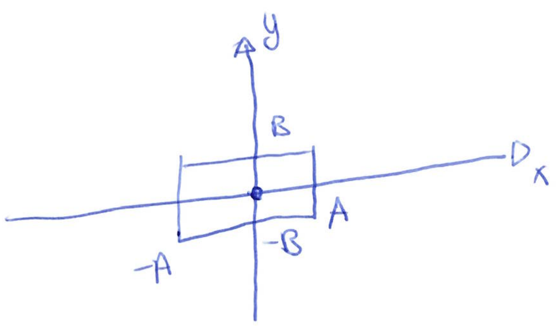
Note: If $\frac{\partial}{\partial y} f(x,y)$ is continuous in D then it is necessarily also bounded.

Example $\quad y' = \frac{1}{2y} \quad \& \quad y(0) = 0$

$$y' = f(x,y) = \frac{1}{2y}$$

$D: \ |x| \le A \quad |y| \le B$

$f(x,y) = \frac{1}{2y}$ is NOT continuous in $y = 0$



$f(x,y)$ is NOT continuous in any rectangular region D centered in $(0,0)$! The hypothesis of the Picard - Lindelöf theorem are not satisfied.

Example $\quad y' = 3y^{2/3} \quad \& \quad y(0)=0$

$$y' = f(x,y) = 3y^{2/3} \qquad y(a)=b$$

$a=0$

$b=0$

$f(x,y) = 3y^{2/3}$ is continuous in D ~~on~~ with

$|x| \le A \quad |y| \le B$ ✓

$$\frac{\partial f}{\partial y} = 3 \cdot \frac{2}{3} y^{-1/3} = 2y^{-1/3}$$ diverges for $y=0$!

$\frac{\partial f}{\partial y}$ cannot be bounded in D

The hypothesis of the Picard-Lindelöf theorem are not satisfied.

Example $y' = 3y^{2/3}$ & $y(0) = 0$

$y' = f(x,y) = 3y^{2/3}$ $y(a) = b$ $a = 0,\ b = 0$

- $f(x,y)$ is continuous in D $|x| \leq A$ $|y| \leq B$

  $f(x,y) = 3y^{2/3}$ ✓

- $\dfrac{\partial f(x,y)}{\partial y} = \dfrac{\partial}{\partial y}(3y^{2/3}) = 2y^{-1/3}$ diverges in $y = 0$!

  X

$\dfrac{\partial f}{\partial y}$ is NOT bounded in any rectangular region D.

The hypothesis of the Picard-Lindelöf theorem are not satisfied!

Example $\frac{dy}{dx} = x^2 |y|^{1/3}$ & $y(0) = 1$

$$\frac{dy}{dx} = f(x,y) = x^2 |y|^{1/3} \qquad y(a) = b$$

$a = 0 \quad b = 1$

① $$f(x,y) = \begin{cases} x^2 y^{1/3} & \text{for } y > 0 \\ x^2 (-y)^{1/3} & \text{for } y \leq 0 \end{cases}$$

D

$|x - 0| \leq A$

$|y - 1| \leq B$

Let us check that it is continuous in $y = 0$

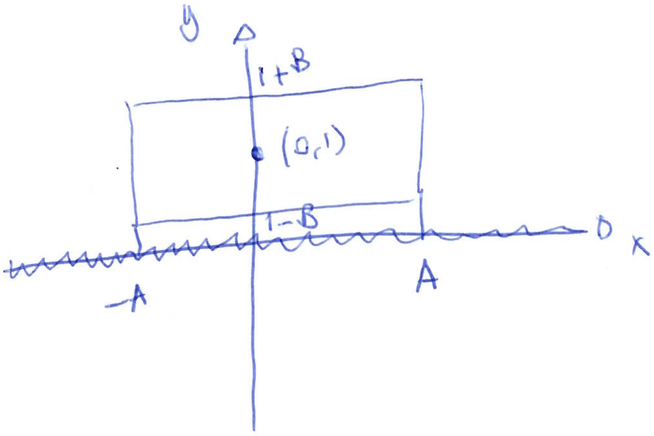
$$\lim_{y \to 0^+} x^2 y^{1/3} = \lim_{y \to 0^-} x^2 (-y)^{1/3} = 0$$

$f(x,y)$ is continuous in D ✓

② Lipschitz condition

$$\frac{\partial f(x,y)}{\partial y} = \begin{cases} x^2 \frac{1}{3} y^{-2/3} & \text{for } y > 0 \\ -x^2 \frac{1}{3} (-y)^{-2/3} & \text{for } y \leq 0 \end{cases}$$

It diverges for $y = 0$!



$-A \le x \le A$

$1-B \le y \le 1+B$

I must impose $\quad 0 < 1-B \quad \boxed{0 < B < 1}$

since for $0 < B < 1$ $\quad \frac{\partial f}{\partial y}$ is bounded in D.

We need to check $\quad A \le \frac{B}{M}$ where $M = \max\limits_{(x,y)\in D} |f(x,y)|$

$$M = \max_{(x,y)\in D} |f(x,y)| = \max_{(x,y)\in D} x^2 |y|^{1/3} = \underbrace{\max_{(x,y)\in D} x^2}_{= A^2} \cdot \underbrace{\max_{(x,y)\in D} |y|^{1/3}}_{(1+B)^{1/3}}$$

$D: \quad |x| \le A \quad |y-1| \le B$

$0 < 1-B \le y \le 1+B$

$M = A^2 (1+B)^{1/3}$

$$A \le \frac{B}{M} = \frac{B}{A^2(1+B)^{1/3}} \Rightarrow A^3 \le \frac{B}{(1+B)^{1/3}}$$

$\boxed{0 < B < 1}$ $\qquad \boxed{0 < A \le \frac{B^{1/3}}{(1+B)^{1/9}}}$