Linear 2^{nd}-order ODEs.

Linear second order ODEs are equations of the type

$$\boxed{a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = f(x)}$$

or equivalenty

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)\,y = f(x)}$$

where $a_2(x), a_1(x), a_0(x)$ and $f(x)$ are continuous functions of $x \in (A,B)$ and $a_2(x) \neq 0$

- If $f(x)=0$ the 2^{nd}-order linear ODE is <u>homogeneous</u>

- If $f(x)\neq 0$ the 2^{nd}-order linear ODE is <u>inhomogeneous</u>

Note that $a_2(x), a_1(x), a_0(x), f(x)$ might be non-linear functions of x

Examples

$3y'' = e^x$ ✓ $a_2(x) = 3$

$a_1(x) = a_0(x) = 0$

$f(x) = e^x$

2^{nd}-order linear ODE
inhomogeneous

$(\tan x)\, y'' + (x+4)^3 y = 0$ 2^{nd}-order linear ODE
homogeneous

$y' + \tanh y = 0$ 1^{st}-order non-linear ODE

$y' + \tanh x = 0$ 1^{st}-order linear inhomogeneous ODE.

The general solution $y_g(x)$ of a linear ODE
is the family of all solutions of the linear ODE.

- For 1^{st}-order ODE the family includes
  1 arbitrary constant

- For 2^{nd}-order ODE the family includes
  2 arbitrary constant.

Linearity of 2^{nd}-order ODEs

We will consider the 2^{nd}-order ODE

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = f(x) \qquad (1)$$

and express the left hand side as

$$\mathcal{L}(y) = a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y$$

Therefo the ODE in (1) can be expressed as

$$\mathcal{L}(y) = f(x) \qquad (2)$$

- If $f(x) = 0$ we have $\mathcal{L}(y) = 0$ homogeneous linear ODE
- If $f(x) \neq 0$ we have $\mathcal{L}(y) = f(x)$ inhomogeneous linear ODE.

Note: This procedure can be performed also for 1^{st}-order linear ODEs by setting $a_2(x) = 0$

Linearity implies that given two arbitrary solutions $y_1(x)$ & $y_2(x)$ which are twice differentiable functions, and given two arbitrary constant $c_1, c_2 \in \mathbb{R}$

$$\mathcal{L}(c_1 y_1(x) + c_2 y_2(x)) = c_1 \mathcal{L}(y_1) + c_2 \mathcal{L}(y_2)$$

Proof Indeed

$$\frac{d}{dx}(c_1 y_1 + c_2 y_2) = c_1 \frac{dy_1}{dx} + c_2 \frac{dy_2}{dx}$$

$$\frac{d^2}{dx^2}(c_1 y + c_2 y_2) = \frac{d}{dx}\left(\frac{d}{dx}(c_1 y_1 + c_2 y_2)\right) =$$

$$= \frac{d}{dx}\left(c_1 \frac{dy_1}{dx} + c_2 \frac{dy_2}{dx}\right) =$$

$$= c_1 \frac{d^2 y_1}{dx^2} + c_2 \frac{d^2 y_2}{dx^2}$$

Therefore $\mathcal{L}(c_1 y_1 + c_2 y_2) = a_2(x) \frac{d^2}{dx^2}(c_1 y_1 + c_2 y_2) +$

$a_1(x) \frac{d}{dx}(c_1 y_1 + c_2 y_2) +$

$a_0(x)(c_1 y_1 + c_2 y_2)$

$$\mathcal{L}(c_1 y_1 + c_2 y_2) = c_1 \underbrace{\left(a_2(x) \frac{d^2 y_1}{dx} + a_1(x) \frac{d y_1}{dx} + a_0 y_1 \right)}_{= \mathcal{L}(y_1)}$$

$$+ c_2 \underbrace{\left(a_2(x) \frac{d^2 y_2}{dx} + a_1(x) \frac{d y_2}{dx} + a_0 y_2 \right)}_{= \mathcal{L}(y_2)}$$

$$\mathcal{L}(c_1 y_1 + c_2 y_2) = c_1 \mathcal{L}(y_1) + c_2 \mathcal{L}(y_2)$$

$\square$

It follows that if y_1 satisfies $\mathcal{L}(y_1) = 0$

if y_2 satisfies $\mathcal{L}(y_2) = 0$

then any linear combination of y_1 an y_2 is also a solution of the homogeneous equation

$$\mathcal{L}(c_1 y_1 + c_2 y_2) = c_1 \underbrace{\mathcal{L}(y_1)}_{=0} + c_2 \underbrace{\mathcal{L}(y_2)}_{=0} = 0$$

Solution of the inhomogeneous linear ODE

We are looking for the general solution of the inhomogeneous linear ODE

$$\mathcal{L}(y) = f(x)$$

that we denote by $y_g(x)$.

Theorem Suppose that $y_p(x)$ is a particular solution of the inhomogeneous ODE

$$\mathcal{L}(y_p) = f(x)$$

Suppose that $y_h(x)$ is the general solution of the homogeneous ODE

$$\mathcal{L}(y_h) = 0$$

Then, the general solutions $y_g(x)$ of the INHOMOGENEOUS linear ODE

$$\mathcal{L}(y) = f(x)$$

can be written as

$$\boxed{y_g(x) = y_p(x) + y_h(x)}$$

Proof Since $y_p(x)$ is a particular solution of the inhomogeneous linear ODE we have

$$\mathcal{L}(y_p(x)) = f(x)$$

Since $y_h(x)$ is the general solution of the ~~in~~homogeneous linear ODE

$$\mathcal{L}(y_h(x)) = 0$$

Therefore $$\mathcal{L}(y_g - y_p) \underset{\substack{\uparrow \\ \text{linearity}}}{=} 1 \cdot \mathcal{L}(y_g) - \mathcal{L}(y_p) = f(x) - f(x) = 0$$

Since $y_g(x)$ is the general solution of the inhomogeneous linear ODE

$$\mathcal{L}(y_g(x)) = f(x)$$

It follows that $y_g - y_p$ is any solution of the homogeneous linear ODE

$$\mathcal{L}(y) = 0 \quad \Longrightarrow \quad y_g(x) - y_p(x) = y_h(x)$$

$$\boxed{y_g(x) = y_p(x) + y_h(x)}$$

$\square$