Scale - invariant ODEs

Scale invariant ODE

$$y' = f\left(\frac{y}{x}\right)$$

They are scale invariant because they are unchanged under the transformation

$$y \to by$$
$$x \to bx$$

Indeed

$$f\left(\frac{y}{x}\right) \to f\left(\frac{\cancel{b}y}{\cancel{b}x}\right) = f\left(\frac{y}{x}\right)$$

$$y' = \frac{dy}{dx} \to \frac{d(by)}{d(bx)} = \frac{\cancel{b}\,dy}{\cancel{b}\,dx} = y'$$

Solution of scale invariant 1^{st}-order ODEs

These are separable upon a change of variable (reducible to separable)

(1) We will denote $z = \frac{y}{x}$ $\Rightarrow y = zx$

We observe that $y' = f\left(\frac{y}{x}\right) = f(z)$ $*$ where $z = z(x)$

(2) Differentiate y with respect to x $y(x) = z(x) \cdot x$

$$y' = z'x + z \cdot 1 = f(z)$$

using $*$

$$z' = \frac{f(z) - z}{x} = g(z)\tilde{f}(x)$$

$g(z) = f(z) - z$

$\tilde{f}(x) = \frac{1}{x}$

1^{st}-order ODE, separable

(3) Solve $\frac{dz}{dx} = \frac{f(z) - z}{x}$ by separation of variable

$z = z(x)$

(4) $y(x) = z(x) \cdot x$

Example

$$y' = -\frac{y}{x} + 1 = f\left(\frac{y}{x}\right) \qquad z = \frac{y}{x}$$

$$f(z) = -z + 1 \qquad y' = f(z)$$

① $z = \frac{y}{x}$ $\qquad y = z \cdot x$

② Differentiate $\qquad y' = \underbrace{z'x + z} = f(z) = \underbrace{-z+1}$

$$z'x + z = -z + 1$$

$$z' = \frac{-2z+1}{x} \qquad \text{separable}$$

③ Solve $\qquad \frac{dz}{dx} = \frac{-2z+1}{x} \Longrightarrow \int \frac{dz}{-2z+1} = \int \frac{dx}{x} + C_1$

RHS: $\qquad H(z) = \int \frac{dz}{-2z+1} = -\frac{1}{2}\ln|2z-1|$

LHS: $\qquad F(x) = \int \frac{dx}{x} = \ln|x|$

$$H(z) = F(x) + C$$

$$-\frac{1}{2}\ln|2z-1| = \ln|x| + C_1 \qquad \text{Implicit solution.}$$

$$\ln|2z-1| = -2\ln|x| - 2C$$

$$|2z-1| = e^{-2\ln|x|-2C} = \frac{1}{|x|^2}\, e^{-2C}$$

$$2z-1 = \pm\frac{1}{x^2}\, e^{-2C} = \underbrace{\boxed{\pm e^{-2C}}}_{=D}\,\frac{1}{x^2}$$

$$\boxed{z = \frac{D}{2x^2} + \frac{1}{2}}$$

Explicit solution for $z = z(x)$

4 $$y = zx = \frac{D}{2x} + \frac{x}{2}$$

General solution

D arbitrary constant

$D \neq 0$

$$\boxed{y = \frac{D}{2x} + \frac{x}{2}}$$

Is $y = \frac{x}{2} - \frac{C}{2x}$ the general solution ? ✓

$C = -D$

Is $y = \frac{C}{x} + \frac{x}{2}$ the general solution ? ✓

$C = \frac{D}{2}$