Boundary Value Problem (BVP)

The boundary value problem (BVP) consists of

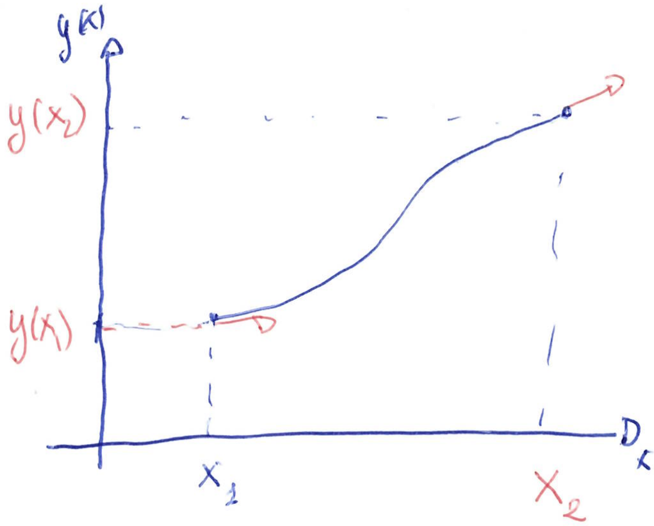
(A) ODE:

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = f(x)$$

$a_2(x) \neq 0$, $a_1(x)$, $a_0(x)$, $f(x)$ continuous for $x \in [x_1, x_2]$

(B) Two bandary conditions specified at the two endpoints

$x = x_1$ and $x = x_2$ with $x_1 < x_2$



We will consider <u>linear bandary conditions</u>

Linear bandary conditions

$$\alpha\, y'(x_1) + \beta\, y(x_1) = b_1$$

$$\gamma\, y'(x_2) + \delta\, y(x_2) = b_2$$

where $\alpha, \beta, \gamma, \delta, b_1, b_2 \in \mathbb{R}$

and

$(\alpha, \beta) \neq (0,0)$

$(\gamma, \delta) \neq (0,0)$

Examples

$$\begin{cases} y(x_1) = b_1 \\ y(x_2) = b_2 \end{cases} \qquad \alpha = 0 \quad \beta = 1 \qquad \gamma = 0 \quad \delta = 1 \qquad \checkmark$$

$$\begin{cases} y'(x_1) = b_1 \\ y'(x_2) = b_2 \end{cases} \qquad \alpha = 1 \quad \beta = 0 \qquad \gamma = 1 \quad \delta = 0$$

$$\begin{cases} 2y'(x_1) + 3y(x_1) = 5 \\ y'(x_2) + 4y(x_2) = 3 \end{cases} \qquad \alpha = 2 \quad \beta = 3 \quad b_1 = 5 \qquad \gamma = 1 \quad \delta = 4 \quad b_2 = 3 \qquad \checkmark$$

$$\begin{cases} y'(x_1) + y(x_2) = 5 \\ y'(x_2) = 5 \end{cases}$$

Is this BCs? No!

Solving BVP

Solving a Boundary Value Problem implies finding the functions (if any) $y(x)$ that satisfy

(A) ODE

(B) the BCs.

in the interval $x \in [x_1; x_2]$

The BVP can have
- no solution
- one solution
- infinite solutions

Homogeneous and Inhomogeneous BCs

A linear boundary condition

$$\alpha y'(x_1) + \beta y(x_1) = b_1$$
$$\gamma y'(x_2) + \delta y(x_2) = b_2$$

is called HOMOGENEOUS if and only if $\begin{cases} b_1 = 0 \\ b_2 = 0 \end{cases}$

A BC that is not homogeneous is called INHOMOGENEOUS

Examples

$$\begin{cases} y(1) + 3y'(1) = 0 \\ y(3) = 3 \end{cases} \qquad \begin{matrix} b_1 = 0 \\ b_2 = 3 \end{matrix} \qquad (*)$$

INHOMOGENEOUS

The corresponding homogeneous BC is obtained by putting

$$\boxed{b_1 = 0 \quad \text{and} \quad b_2 = 0}$$

The corresponding homogeneous BC of $(*)$ is

$$\begin{cases} y(1) + 3y'(1) = 0 \\ y(3) = 0 \end{cases}$$

Homogeneous and Inhomogeneous BVP

Let us consider the BVP

(A) ODE:

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = f(x)$$

$a_2(x) \neq 0$, $a_1(x)$, $a_0(x)$, $f(x)$ continuous for $x \in [x_1, x_2]$

(B) linear BCs

$$\alpha y'(x_1) + \beta y(x_1) = b_1 \qquad (\alpha, \beta) \neq (0,0)$$

$$\gamma y'(x_2) + \delta y(x_2) = b_2 \qquad (\gamma, \delta) \neq (0,0)$$

$$\alpha, \beta, \gamma, \delta, b_1, b_2 \in \mathbb{R}$$

The BVP is HOMOGENEOUS if and only if

(A) the ODE is homogeneous $\quad f(x) = 0$

(B) the BCs are homogeneous $\quad b_1 = b_2 = 0$

The BVP that is not homogeneous is called

INHOMOGENEOUS

Examples

ODE: $y'' + y = e^x$ $\qquad f(x) = e^x$

BC: $\begin{cases} y(0) = 1 \\ y'(2) = 3 \end{cases}$ $\qquad b_1 = 1$, $b_2 = 3$

BVP is INHOMOGENEOUS

ODE $e^x y'' + \sin x \, y' = 0$ $\qquad f(x) = 0$

BC: $\begin{cases} y(1) = 0 \\ y(2) = 3 \end{cases}$ $\qquad (**)$

BVP is INHOMOGENEOUS

ODE: $e^x y'' + \sin x \, y' = 0$

BC $\begin{cases} y(1) = 0 \\ y(2) = 0 \end{cases}$

The BVP is HOMOGENEOUS.

This BVP is the corresponding homogeneous BVP of $(**)$