The Theorem of the Alternative

The Boundary Value Problem (BVP) for 2^{nd}-order linear ODEs with linear boundary conditions comprises of

(A) ODE:

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = f(x)$$

~~when~~ with $a_2(x) \neq 0$, $a_1(x)$, $a_0(x)$, $f(x)$ ~~are~~ continuous in $x \in [x_1, x_2]$

(B) Linear BCs:

$$\alpha y'(x_1) + \beta y(x_1) = b_1$$

$$\gamma y'(x_2) + \delta y(x_2) = b_2$$

with $\alpha, \beta, \gamma, \delta, b_1, b_2 \in \mathbb{R}$

$(\alpha, \beta) \neq (0,0)$

$(\gamma, \delta) \neq (0,0)$

Theorem of the Alternative

Consider the BVP for 2^{nd}-order linear ODEs with linear BCs.

Only two alternatives are possible

1. Either the BVP has a UNIQUE solution for every choice of $f(x)$, b_1, b_2

2. Or the corresponding homogeneous BVP has INFINITE solution

   AND the inhomogeneous BVP has

   (i) INFINITE MANY SOLUTIONS for some choices of $f(x)$, b_1 and b_2

   (ii) NO SOLUTIONS for some choice of $f(x)$ b_1, b_2

Applications of the Theorem

- An inhomogeneous BVP has a UNIQUE SOLUTION

  if and only if

  the corresponding homogeneous ~~problem~~ BVP has a

  UNIQUE (trivial) solution.

- If the corresponding homogeneous BVP has INFINITE solutions

  the inhomogeneous BVP can either have

  INFINITE solutions

  NO SOLUTIONS

  (you need to check which option applies)

Example For which value of $b>0$ the following BVP has a UNIQUE solution?

$$y'' + b^2 y = \sin x \qquad \begin{cases} y(0) = 3 \\ y(1) = -2 \end{cases}$$

$b>0$

This is an inhomogeneous BVP

For the Theorem of the Alternative this BVP has a UNIQUE solution if and only if the corresponding homogeneous BVP has a UNIQUE solution.

Corresponding homogeneous BVP.

$$y'' + b^2 y = 0 \qquad \begin{cases} y(0) = 0 \\ y(1) = 0 \end{cases} \quad (*)$$

with $b>0$

Let us find the value of $b>0$ such that $(*)$ has a UNIQUE solution.

(1) Solve the ODE $y'' + b^2 y = 0$

Characteristic equation $\qquad$ Roots

$$M_2(\lambda) = \lambda^2 + b^2 = 0 \implies \lambda^2 = -b^2 \implies \begin{matrix} \lambda_1 = ib \\ \lambda_2 = -ib \end{matrix}$$

General solution

$$\boxed{y_g(x) = A\cos bx + B\sin bx}$$

where $A, B \in \mathbb{R}$ are arbitrary constants

(2) Impose the BCs
$$\begin{cases} y(0)=0 \\ y(1)=0 \end{cases}$$

$$0 = y(0) = A\cos b\cdot 0 + B\sin b\cdot 0 = A \quad \Rightarrow \boxed{A=0}$$

$$0 = y(1) = \underset{\overset{\parallel}{0}}{A}\cos b + B\sin b = B\sin b \quad \Rightarrow \boxed{B\sin b = 0}$$

We have $A=0$ and $B\sin b = 0$

$B\sin b = 0$ implies either i) $B=0$ $\sin b \neq 0$

ii) $\sin b = 0$, B is arbitrary.

In the case i) we have a UNIQUE solution

ii) we have INFINITE solution

We have a UNIQUE solution if and only if $\sin b \neq 0$

$b \neq n\pi$ where $n \in \mathbb{N}$

$\Rightarrow$ It follows that the inhomogeneous BVP

$$y'' + b^2 y = \sin x \qquad \begin{cases} y(0)=3 \\ y(1)=-2 \end{cases}$$

has a UNIQUE solution for every value of $b>0$ such that $b \neq n\pi$ with $n \in \mathbb{N}$

- Find the smallest $b>0$ such that the BVP

$$y''+b^2y=0 \qquad \begin{cases} y(0)=5 \\ y(1)=-5 \end{cases} \qquad (**)$$

has NO SOLUTIONS.

(1) An inhomogeneous BVP has no solutions ~~if and only if~~ only if its corresponding homogeneous BVP has infinite solutions.

The homogeneous BVP

$$y''+b^2y=0 \qquad \begin{cases} y(0)=0 \\ y(1)=0 \end{cases}$$

has infinite solutions if and only if $b=n\pi$ with $n\in\mathbb{N}$ (see previous example).

$$\boxed{\text{For the Theorem of the Alternative if } b=n\pi \text{ with } n\in\mathbb{N} \text{ the inhomogeneous BVP } (**) \text{ has either no solution or infinite many solutions.}}$$

We need to check directly when it has no solution by going through the values

$$b=\pi,\ 2\pi,\ 3\pi\ldots$$

(A) We check $b = \pi$

The general solution is

$$y_g(x) = A\cos \pi x + B \sin \pi x$$ with $A, B \in \mathbb{R}$ arbitrary constants

Imposing $\begin{cases} y(0) = 5 \\ y(1) = -5 \end{cases}$

$$5 = y(0) = A\cos 0 + B \sin 0 \quad \Rightarrow \boxed{A = 5}$$

$$-5 = y(1) = A \underbrace{\cos \pi}_{-1} + B \underbrace{\sin \pi}_{0} \quad \Rightarrow \boxed{A = 5}$$

B can be arbitrary! the BVP has infinite solutions

$$y_g(x) = 5\cos \pi x + B \sin \pi x$$ with $B \in \mathbb{R}$ and arbitrary.

(B) We need to check $b = 2\pi$.

The general solution is $y_g(x) = A\cos 2\pi x + B \sin 2\pi x$

$A, B \in \mathbb{R}$ and arbitrary.

Imposing $\begin{cases} y(0) = 5 \\ y(1) = -5 \end{cases}$

$$5 = y(0) = A\cos 2\pi \cdot 0 + B \sin 2\pi \cdot 0 = A \qquad \boxed{A = 5}$$

$$-5 = y(1) = A\cos 2\pi + B \sin 2\pi = +A \qquad \boxed{A = -5}$$

The BVP has No solution

The smallest value of $b>0$ for which BVP (**) has

no solution is $\boxed{b = 2\pi}$